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Almost-Periodic Solutions in Various Metrics of Higher-Order Differential Equations with a Nonlinear Restoring Term

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Abstract

Almost-periodic solutions in various metrics (Stepanov, Weyl, Besicovitch) of higher-order differential equations with a nonlinear Lipschitz-continuous restoring term are investigated. The main emphasis is focused on a Lipschitz constant which is the same as for uniformly almost-periodic solutions treated in [A1] and much better than those from our investigations for differential systems in [A2], [A3], [AB], [ABL], [AK]. The upper estimates of ε for ε-almost-periods of solutions and their derivatives are also deduced under various restrictions imposed on the constant coefficients of the linear differential operator on the left-hand side of the given equation. Besides the existence, uniqueness and localization of almost-periodic solutions and their derivatives are established.

Key words: Almost-periodic solutions; various metrics; higher-order differential equation; nonlinear restoring term; existence and uniqueness criteria.

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1 Introduction

We shall consider the differential equation

$$y^{(n)} + \sum_{j=1}^{n} a_j y^{(n-j)} = f(y) + p(t), \tag{1}$$

where $a_j \in \mathbb{R}$, $j = 1, \dots, n$, are real constants such that the real parts of the roots of the characteristic polynomial associated with the linear operator on the left-hand side of (1), namely

$$\lambda^n + \sum_{j=1}^{n} a_j \lambda^{n-j}, \tag{2}$$

are at least nonzero, i.e. $\mathrm{Re}\,\lambda_j \neq 0$, $j = 1, \dots, n$. It is well-known that the related Routh–Hurwitz conditions are necessary and sufficient for $\mathrm{Re}\,\lambda_j < 0$, $j = 1, \dots, n$, i.e. in order polynomial (2) to be stable. In this case, all coefficients $a_j \in \mathbb{R}$ in (1) must be positive, i.e. $a_j > 0$, $j = 1, \dots, n$. One can also find necessary and sufficient conditions in order all roots of (2) to be negative, but for characteristic polynomials of a higher degree these conditions are rather cumbersome (see e.g. [AG, Chapter III.5]). Assume, furthermore, that the restoring term $f \in \mathrm{Lip}(\mathbb{R}, \mathbb{R})$ is a bounded Lipschitz-continuous function with constant $L < |a_n|$, and that the forcing term $p \in \mathrm{L}^1_{loc}(\mathbb{R}, \mathbb{R})$ is an essentially bounded, locally Lebesgue integrable function which will be successively supposed to be almost-periodic (a.p.) in the sense of Stepanov, Weyl or Besicovitch.

The main aim of the present paper is to extend appropriately sufficient conditions for the existence of uniformly almost-periodic solutions and their derivatives, obtained for (1) in [A1] (cf. also [AG, Chapter III.10]), provided the forcing term p is almost-periodic in a more general sense (Stepanov, Weyl, Besicovitch). Although the existence criteria for such a.p. solutions and their derivatives can be deduced from our earlier results for differential systems, namely for Stepanov a.p. solutions in [AB], for Weyl a.p. solutions in [A2], [A3], and for Besicovitch a.p. solutions in [ABL] (cf. also [AG, Chapter III.10]), the upper estimates for Lipschitz constant L related to f would be very rough (cf. e.g. [AK]). Another purpose therefore consists in obtaining much sharper inequality for L, namely $L < |a_n|$. Since this is possible only if the roots of (2) are at least nonzero real (otherwise, the desired estimates for L would explicitly depend on them), we shall still assume that the coefficients a_j, $j = 1, \dots, n$, yield nonzero real roots.

Higher-order differential equations of the type (1), where $n > 2$, have not been treated w.r.t. the existence of a.p. solutions so often (see e.g. [Kh], [KBK], [L]). The investigations of the other authors of more general than uniformly a.p. solutions were also quite rare (see e.g. [BFSD1]–[BFSD3], [BFH], [DHS], [DM], [H], [Ku], [LZ], [P], [ZL]). As far as we know, apart from our mentioned papers [A2], [A3], [AB], [ABL] and [LZ], [P], [R], [ZL], almost-periodic solutions in the generalized sense of (1), where $n > 2$, have not yet been studied with the indicated respect.

The paper is organized as follows. After some preliminaries, the main existence results are formulated. Roughly speaking, as much as we impose on the coefficients a_j, $j = 1, \ldots, n$, on the left-hand side of (1), as good estimates of ε for ε-almost-periods of a.p. solutions and their derivatives we obtain. Moreover, more transparent estimates of (entirely bounded) a.p. solutions allow us to replace global boundedness assumption on f by restrictions localized only on certain domains. This will be done, besides another, in concluding remarks, jointly with extending our results to differential inclusions, on the basis of selection theorems in [HP] and [D1]–[D3], [DS].

2 Some preliminaries

At first, we recall various types of almost-periodicity.

Definition 1 Let us introduce the following (pseudo-) metrics:
(Stepanov)

$$D_{S_l}(f,g) := \sup_{a\in\mathbb{R}} \frac{1}{l} \int_a^{a+l} |f(t) - g(t)|\, dt,$$

(Weyl)

$$D_W(f,g) := \lim_{l\to\infty} \sup_{a\in\mathbb{R}} \frac{1}{l} \int_a^{a+l} |f(t) - g(t)|\, dt = \lim_{l\to\infty} D_{S_l}(f,g),$$

(Besicovitch)

$$D_B(f,g) := \limsup_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(t) - g(t)|\, dt,$$

where $f, g : \mathbb{R} \to \mathbb{R}$ are measurable functions. Denoting by D_G any of the above (pseudo-) metrics, by the metric space (G, D_G), we understand the related quotient space in the sense that we identify such elements f_1, f_2, for which $D_G(f_1, f_2) = 0$.

Definition 2 A function $f \in \mathrm{L}^1_{loc}(\mathbb{R},\mathbb{R})$ is said to be *G-almost-periodic* (G-a.p.) if

$$\forall\varepsilon > 0\ \exists k > 0\ \forall a \in \mathbb{R}\ \exists\tau \in [a, a+k] : D_G(f(t+\tau), f(t)) < \varepsilon.$$

The above τ is called an *ε-almost-period* in the respective sense.

Instead of D_{S_1}-a.p. or D_W-a.p. or D_B-a.p. function, we shall write S_1-a.p. or W-a.p. or B-a.p., respectively.

The following definition uses curiously the Stepanov metric for the almost-periodicity in the sense of H. Weyl.

Definition 3 A function $f \in \mathrm{L}^1_{loc}(\mathbb{R}, \mathbb{R})$ is said to be *equi-Weyl-almost-periodic* (equi-W-a.p.) if

$$\forall \varepsilon > 0 \ \exists k, l_0(\varepsilon) > 0 \ \forall a \in \mathbb{R} \ \exists \tau \in [a, a+k] : \\ D_{S_l}(f(t+\tau), f(t)) < \varepsilon, \quad \forall l \geq l_0(\varepsilon).$$

Remark 1 It is well-known (see e.g. [ABG], [L], [LZ]) that, without any loss of generality, we can take $l_0 \geq 1$ in Definition 3.

Definition 4 A function $f : \mathbb{R} \to \mathbb{R}$ is called *uniformly G-continuous* if

$$\forall \varepsilon > 0 \ \exists \delta = \delta(\varepsilon) > 0 : |h| < \delta \Longrightarrow D_G(f(t+h), f(t)) < \varepsilon.$$

If, in particular, the above implication holds for a continuous function f with D_G replaced by the sup-norm, then we simply speak about *uniform continuity* of f.

In the following sections, the existence of almost-periodic solutions and their derivatives in various metrics will be proved by three different techniques for differential equation (1).

Hence, consider the differential equation (1), i.e.

$$y^{(n)} + \sum_{j=1}^{n} a_j y^{(n-j)} = f(y) + p(t),$$

where $a_j \in \mathbb{R}$, $j = 1, \dots n$, $f \in \mathrm{Lip}(\mathbb{R}, \mathbb{R})$ and $p \in \mathrm{L}^1_{loc}(\mathbb{R}, \mathbb{R})$.

Assume, furthermore, that

(i) all roots λ_j, $j = 1, \dots n$, of the characteristic polynomial (2), i.e. of

$$\lambda^n + \sum_{j=1}^{n} a_j \lambda^{n-j},$$

are nonzero and real;

(ii) f is bounded and Lipschitz on $\mathbb{R}$, i.e. there exists $L > 0$ such that

$$|f(x) - f(y)| \leq L|x - y|, \qquad \forall x, y \in \mathbb{R};$$

(iii) p is an essentially bounded $\bar{G}$-a.p. function, where $\bar{G}$ means either S or W or B or equi-W case;

(iv) there exists a positive constant D_0 s.t.

$$\operatorname*{supess}_{t \in \mathbb{R}} |p(t)| + \sup_{y \in \mathbb{R}} |f(y)| \leq D_0.$$

In the entire text, by a *solution* $y(\cdot)$ of (1), we shall mean the one in the sense of *Carathéodory*, i.e. such that $y^{(n-1)}(\cdot)$ is locally absolutely continuous.

The following lemma guarantees the existence of a unique bounded solution of (1), including its suitable representation for our application, and the same for its derivatives.

Lemma 1 *Assume that all roots of the characteristic polynomial (2) are nonzero and real (i.e. (i)). Under the assumption (iv), and (ii) with* $L < |a_n|$*, equation (1) has exactly one (Carathéodory) entirely bounded solution* $y(\cdot)$ *given by the formula*

$$y(t) = \int_{\Lambda_1}^{t} \int_{\Lambda_2}^{t_1} \dots \int_{\Lambda_n}^{t_{n-1}} e^{\lambda_1 t + (\lambda_2 - \lambda_1) t_1 + \ldots + (\lambda_n - \lambda_{n-1}) t_{n-1} - \lambda_n t_n} [f(y(t_n)) + p(t_n)] \, dt_n \dots dt_1,$$

where $\Lambda_j = +\infty \cdot \lambda_j$*,* $j = 1, \dots, n$*.*

Denoting the right-hand side of the preceding formula by $[1, \dots, n]$*, the* k*-th derivatives (*$k = 1, \dots, n-1$*) of solution* $y(\cdot)$ *satisfy*

$$\begin{aligned} y^{(k)}(t) = \frac{d^k([1, \dots, n])}{dt^k} &= [k+1, \dots, n] + \sum_{c_1=1}^{k} \lambda_{c_1} [c_1, k+1, \dots, n] \\ &+ \sum_{\substack{c_1, c_2 = 1 \\ c_1 < c_2}}^{k} \lambda_{c_1} \lambda_{c_2} [c_1, c_2, k+1, \dots, n] + \dots \\ + \sum_{\substack{c_1, \dots, c_p = 1 \\ c_1 < \dots < c_p}}^{k} \left(\prod_{i=1}^{p} \lambda_{c_i} \right) &[c_1, \dots, c_p, k+1, \dots, n] + \dots + \left(\prod_{i=1}^{k} \lambda_i \right) [1, \dots, n], \end{aligned}$$

where

$$[c, \dots, n] = \int_{\Lambda_c}^{t} \int_{\Lambda_{c+1}}^{t_c} \dots \int_{\Lambda_n}^{t_{n-1}} e^{\lambda_c t + (\lambda_{c+1} - \lambda_c) t_c + \ldots + (\lambda_n - \lambda_{n-1}) t_{n-1} - \lambda_n t_n} \times [f(y(t_n)) + p(t_n)] \, dt_n \dots dt_{c+1} \, dt_c.$$

Proof The complete proof can be found in [AG]. The existence of a bounded solution is verified at page 554 (cf. also pp. 329–330). The representation formula is given at p. 321 (Lemma 5.45) and the formula for the k-th derivative is derived at pp. 324–325 (Lemma 5.61). The uniqueness is proved at p. 556. □

Remark 2 The solution $y(\cdot)$ in Lemma 1 satisfies

$$\sup_{t \in \mathbb{R}} |y(t)| \le \frac{D_0}{|a_n|}$$

(see [AG, p. 323]) and its k-th derivative ($k = 1, \dots, n-1$) can be estimated by

a) $\sup_{t \in \mathbb{R}} |y^{(k)}(t)| \le \frac{2^k D_0}{|a_n|} \prod_{j=1}^{k} |\alpha_j|$, when the characteristic polynomial has only real nonzero roots (see [AG, Lemma 5.63 at pp. 325–326]);

b) $\sup_{t\in\mathbb{R}} |y^{(k)}(t)| \le \dfrac{2^k D_0}{|a_{n-k}|}$, provided each of the shifted polynomials

$$\lambda^{n-p} + \sum_{j=1}^{n-p} a_j \lambda^{n-p-j}, \quad p = 0, \dots, n-1,$$

admits real nonzero roots (see [AG, Lemma 5.70 at p. 327]);

c) $\sup_{t\in\mathbb{R}} |y^{(k)}(t)| \le \dfrac{2^k a_k D_0}{\binom{n}{k} a_n}$, whenever all roots of the characteristic polynomial are negative (see [AG, Lemma 5.67 at p. 326]).

The meaning of constant D_0 can be seen in (iv).

Moreover, the estimates for the k-th derivatives are independent of the permutation of the roots (see [AG, p. 326]).

Remark 3 Observe that, under the assumptions (i), (iv), a bounded solution of (1) with its derivatives, up to the $(n-1)$-th order, are uniformly continuous, and subsequently also uniformly G-continuous.

Remark 4 The existence and representation parts of Lemma 1 are true if only the real parts of roots of (2) are assumed to be nonzero (cf. [AG, Chapter III.5]). On the other hand, the related estimates for solutions $y(\cdot)$ and their derivatives $y^{(k)}(\cdot)$, $k = 1, \dots, n-1$, do not depend explicitly on the coefficients a_k, but only on the real parts of the roots of (2) (cf. again [AG, Chapter III.5]).

3 Existence of a.p. solutions: case of nonzero real roots

The following main theorem is stated under the most general assumptions, when comparing with other main results of this paper.

Theorem 1 *Let the above conditions (i)–(iv) be satisfied. If $L < |a_n|$, then equation (1) admits a unique bounded $\bar{G}$-a.p. solution with bounded $\bar{G}$-a.p. derivatives, up to the $(n-1)$-th order.*

Moreover, the ε-almost-period of $p(\cdot)$ implies the $\frac{1}{|a_n|-L}$ ε-almost-period of the solution $y(\cdot)$ and the $\frac{2^k|\lambda_1\dots\lambda_k|}{|a_n|-L}$ ε-almost-period of the k-th derivative $y^{(k)}(\cdot)$ of the solution in the $\bar{G}$-(pseudo-)metric, for $k = 1, \dots, n-1$, where $\lambda_1, \dots, \lambda_n$ are the roots of the characteristic polynomial $\lambda^n + \sum_{j=1}^{n} a_j \lambda^{n-j}$.

Proof It follows from Lemma 1 that equation (1) admits a unique bounded solution of the form as above. Using the appropriate representation of this solution, one can obtain by means of (ii):

$$|y(t+\tau) - y(t)| \le$$

$$\le \left| \int_{\Lambda_1}^{t} \int_{\Lambda_2}^{t_1} \dots \int_{\Lambda_n}^{t_{n-1}} e^{\lambda_1 t + (\lambda_2 - \lambda_1) t_1 + \dots + (\lambda_n - \lambda_{n-1}) t_{n-1} - \lambda_n t_n} \left[|f(y(t_n+\tau)) - f(y(t_n))| \right.\right.$$

$$+\,|p(t_n+\tau)-p(t_n)|\,]\;dt_n\,dt_{n-1}\dots dt_1\Bigg|$$

$$\le \Bigg|\int_{\Lambda_1}^{t}\int_{\Lambda_2}^{t_1}\dots\int_{\Lambda_n}^{t_{n-1}} e^{\lambda_1 t+(\lambda_2-\lambda_1)t_1+\ldots+(\lambda_n-\lambda_{n-1})t_{n-1}-\lambda_n t_n}\,(L|y(t_n+\tau)-y(t_n)|$$

$$+\,|p(t_n+\tau)-p(t_n)|\,)\,dt_n\,dt_{n-1}\dots dt_1\Bigg|$$

$$=\Big|(-\frac{1}{\lambda_n})\dots(-\frac{1}{\lambda_1})\Big|\int_0^1\int_0^1\dots\int_0^1 L\Big|y(-\sum_{j=1}^{n}\frac{\ln s_j}{\lambda_j}+t+\tau)-y(-\sum_{j=1}^{n}\frac{\ln s_j}{\lambda_j}+t)\Big|$$

$$+\Big|p(-\sum_{j=1}^{n}\frac{\ln s_j}{\lambda_j}+t+\tau)-p(-\sum_{j=1}^{n}\frac{\ln s_j}{\lambda_j}+t)\Big|\,ds_n\dots ds_2\,ds_1,$$

where the last equality can be obtained by virtue of successive substitutions $s_j=e^{\lambda_j(t_{j-1}-t_j)}$, for $j=n,n-1\dots,2$, and $s_1=e^{\lambda_1(t-t_1)}$.

Now, we shall prove the $\bar{G}$-almost-periodicity of solution $y(\cdot)$, when applying assumption (iii). To employ all of the considered (pseudo-) metrics, we will need the following estimate (for $a<b$, $a,b\in\mathbb{R}$):

$$\int_a^b |y(t+\tau)-y(t)|\,dt\le$$

$$\le\frac{1}{|\lambda_n\cdot\ldots\cdot\lambda_1|}\int_a^b\int_0^1\dots\int_0^1 L\Big|y(-\sum_{j=1}^{n}\frac{\ln s_j}{\lambda_j}+t+\tau)-y(-\sum_{j=1}^{n}\frac{\ln s_j}{\lambda_j}+t)\Big|$$

$$+\Big|p(-\sum_{j=1}^{n}\frac{\ln s_j}{\lambda_j}+t+\tau)-p(-\sum_{j=1}^{n}\frac{\ln s_j}{\lambda_j}+t)\Big|\,ds_n\dots ds_2\,ds_1\,dt$$

$$=\frac{L}{|a_n|}\int_0^1\dots\int_0^1\int_a^b\Big|y(-\sum_{j=1}^{n}\frac{\ln s_j}{\lambda_j}+t+\tau)-y(-\sum_{j=1}^{n}\frac{\ln s_j}{\lambda_j}+t)\Big|\,dt\,ds_n\dots ds_1$$

$$+\frac{1}{|a_n|}\int_0^1\dots\int_0^1\int_a^b\Big|p(-\sum_{j=1}^{n}\frac{\ln s_j}{\lambda_j}+t+\tau)-p(-\sum_{j=1}^{n}\frac{\ln s_j}{\lambda_j}+t)\Big|\,dt\,ds_n\dots ds_1.$$

Using the Stepanov metric, we get ($a:=u$, $b:=u+1$):

$$\sup_{u\in\mathbb{R}}\int_u^{u+1}|\,y(t+\tau)-y(t)\,|\;dt\le$$

$$\leq \frac{L}{|a_n|} \sup_{u\in\mathbb{R}} \int_0^1 \dots \int_0^1 \int_u^{u+1} \Big| y(-\sum_{j=1}^n \frac{\ln s_j}{\lambda_j} + t + \tau) - y(-\sum_{j=1}^n \frac{\ln s_j}{\lambda_j} + t) \Big| \, dt \, ds_n \dots ds_1$$

$$+ \frac{1}{|a_n|} \sup_{u\in\mathbb{R}} \int_0^1 \dots \int_0^1 \int_u^{u+1} \Big| p(-\sum_{j=1}^n \frac{\ln s_j}{\lambda_j} + t + \tau) - p(-\sum_{j=1}^n \frac{\ln s_j}{\lambda_j} + t) \Big| \, dt \, ds_n \dots ds_1$$

$$< \frac{L}{|a_n|} \sup_{u\in\mathbb{R}} \int_u^{u+1} |y(t+\tau) - y(t)| \, dt + \frac{\varepsilon}{|a_n|} \int_0^1 \dots \int_0^1 ds_n \dots ds_1.$$

Hence,

$$\sup_{u\in\mathbb{R}} \int_u^{u+1} |y(t+\tau) - y(t)| \, dt < \frac{\varepsilon}{|a_n| - L} = \widehat{\varepsilon}.$$

Thus, under the assumption $|a_n| > L$, the $\widehat{\varepsilon}$-almost period of solution $y(\cdot)$ corresponds to an ε-almost period of function $p(\cdot)$ (in the sense of Stepanov).

For the equi-Weyl case, we get ($a := u$, $b := u + l$):

$$\sup_{u\in\mathbb{R}} \frac{1}{l} \int_u^{u+l} |y(t+\tau) - y(t)| \, dt \leq$$

$$\leq \frac{L}{|a_n|} \sup_{u\in\mathbb{R}} \int_0^1 \dots \int_0^1 \frac{1}{l} \int_u^{u+l} \Big| y(-\sum_{j=1}^n \frac{\ln s_j}{\lambda_j} + t + \tau) - y(-\sum_{j=1}^n \frac{\ln s_j}{\lambda_j} + t) \Big| \, dt \, ds_n \dots ds_1$$

$$+ \frac{1}{|a_n|} \sup_{u\in\mathbb{R}} \int_0^1 \dots \int_0^1 \frac{1}{l} \int_u^{u+l} \Big| p(-\sum_{j=1}^n \frac{\ln s_j}{\lambda_j} + t + \tau) - p(-\sum_{j=1}^n \frac{\ln s_j}{\lambda_j} + t) \Big| \, dt \, ds_n \dots ds_1$$

$$< \frac{L}{|a_n|} \sup_{u\in\mathbb{R}} \frac{1}{l} \int_u^{u+l} |y(t+\tau) - y(t)| \, dt + \frac{\varepsilon}{|a_n|} \int_0^1 \dots \int_0^1 ds_n \dots ds_1,$$

which implies

$$\sup_{u\in\mathbb{R}} \frac{1}{l} \int_u^{u+l} |y(t+\tau) - y(t)| \, dt < \frac{\varepsilon}{|a_n| - L} = \widehat{\varepsilon}, \qquad \forall l \geq l_0.$$

By the above estimate, we can also obtain the following inequalities for the W-almost-periodicity:

$$\lim_{l\to\infty} \Big[\sup_{u\in\mathbb{R}} \frac{1}{l} \int_u^{u+l} |y(t+\tau) - y(t)| \, dt \Big] \leq$$

$$\leq \frac{L}{|a_n|} \lim_{l\to\infty} \sup_{u\in\mathbb{R}} \int_0^1 \dots \int_0^1 \frac{1}{l} \int_u^{u+l} \Big| y(-\sum_{j=1}^{n} \frac{\ln s_j}{\lambda_j} + t + \tau)$$

$$- y(-\sum_{j=1}^{n} \frac{\ln s_j}{\lambda_j} + t) \Big| \, dt \, ds_n \dots ds_1$$

$$+ \frac{1}{|a_n|} \lim_{l\to\infty} \sup_{u\in\mathbb{R}} \int_0^1 \dots \int_0^1 \frac{1}{l} \int_u^{u+l} \Big| p(-\sum_{j=1}^{n} \frac{\ln s_j}{\lambda_j} + t + \tau)$$

$$- p(-\sum_{j=1}^{n} \frac{\ln s_j}{\lambda_j} + t) \Big| \, dt \, ds_n \dots ds_1$$

$$< \frac{L}{|a_n|} \lim_{l\to\infty} \sup_{u\in\mathbb{R}} \frac{1}{l} \int_u^{u+l} |y(t+\tau) - y(t)| \, dt + \frac{\varepsilon}{|a_n|} \int_0^1 \dots \int_0^1 ds_n \dots ds_1 .$$

Thus,

$$\lim_{l\to\infty} \left[\sup_{u\in\mathbb{R}} \frac{1}{l} \int_u^{u+l} |\, y(t+\tau) - y(t) \,| \; dt \right] < \frac{\varepsilon}{|a_n| - L} = \widehat{\varepsilon}$$

holds for the W-almost-periodicity of $y(\cdot)$.

The proof for B-almost-periodicity is again based on the application of the inequality derived above. Hence, ($a := -T$, $b := T$):

$$\limsup_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |y(t+\tau) - y(t)| \, dt \leq$$

$$\leq \frac{L}{|a_n|} \limsup_{T\to\infty} \frac{1}{2T} \int_0^1 \dots \int_0^1 \int_{-T}^{T} \Big| y(-\sum_{j=1}^{n} \frac{\ln s_j}{\lambda_j} + t + \tau)$$

$$- y(-\sum_{j=1}^{n} \frac{\ln s_j}{\lambda_j} + t) \Big| \, dt \, ds_n \dots ds_1$$

$$+ \frac{1}{|a_n|} \limsup_{T\to\infty} \frac{1}{2T} \int_0^1 \dots \int_0^1 \int_{-T}^{T} \Big| p(-\sum_{j=1}^{n} \frac{\ln s_j}{\lambda_j} + t + \tau)$$

$$- p(-\sum_{j=1}^{n} \frac{\ln s_j}{\lambda_j} + t) \Big| \, dt \, ds_n \dots ds_1$$

$$< \frac{L}{|a_n|} \limsup_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |y(t+\tau) - y(t)| \, dt + \frac{\varepsilon}{|a_n|} \int_0^1 \dots \int_0^1 ds_n \dots ds_1 .$$

Repeating the procedure as in the preceding cases, one arrives at

$$\limsup_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |y(t+\tau) - y(t)|\, dt < \frac{\varepsilon}{|a_n| - L} = \widehat{\varepsilon}.$$

We could see that the almost-periodicity of solution $y(\cdot)$ was verified in all given (pseudo-)metrics, whenever $L < |a_n|$. Moreover, to ε-almost period of $p(\cdot)$, there corresponds the $\frac{\varepsilon}{|a_n|-L}$-almost period of solution $y(\cdot)$ (in the related pseudo-metric).

To prove the $\bar{G}$-almost-periodicity of the derivatives $y^{(k)}(\cdot)$, we use the formula from Lemma 1. Hence, applying (ii) and making successive substitutions as in the preceding part of the proof, we get

$$|y^{(k)}(t+\tau) - y^{(k)}(t)| \leq$$

$$\leq \Bigg| \int_{\Lambda_{k+1}}^{t} \int_{\Lambda_{k+2}}^{t_{k+1}} \dots \int_{\Lambda_n}^{t_{n-1}} e^{\lambda_{k+1}t + (\lambda_{k+2}-\lambda_{k+1})t_{k+1} + \dots + (\lambda_n - \lambda_{n-1})t_{n-1} - \lambda_n t_n}$$

$$\times \left[|f(y(t_n+\tau)) - f(y(t_n))| + |p(t_n+\tau) - p(t_n)| \right] dt_n \dots dt_{k+1} \Bigg|$$

$$+ \sum_{j=1}^{k} \Bigg| \lambda_j \int_{\Lambda_j}^{t} \int_{\Lambda_{k+1}}^{t_j} \dots \int_{\Lambda_n}^{t_{n-1}} e^{\lambda_1 t + (\lambda_2 - \lambda_1)t_1 + \dots + (\lambda_n - \lambda_{n-1})t_{n-1} - \lambda_n t_n}$$

$$\times \left[|f(y(t_n+\tau)) - f(y(t_n))| + |p(t_n+\tau) - p(t_n)| \right] dt_n \dots dt_j \Bigg|$$

$$+ \sum_{\substack{i,j=1 \\ i<j}}^{k} \Bigg| \lambda_i \lambda_j \int_{\Lambda_i}^{t} \int_{\Lambda_j}^{t_i} \int_{\Lambda_{k+1}}^{t_j} \dots \int_{\Lambda_n}^{t_{n-1}} e^{\lambda_i t + (\lambda_j - \lambda_i)t_i + \dots + (\lambda_n - \lambda_{n-1})t_{n-1} - \lambda_n t_n}$$

$$\times \left[|f(y(t_n+\tau)) - f(y(t_n))| + |p(t_n+\tau) - p(t_n)| \right] dt_n \dots dt_j\, dt_i \Bigg| + \dots$$

$$+ \Bigg| \left(\prod_{j=1}^{k} \lambda_j \right) \int_{\Lambda_1}^{t} \int_{\Lambda_2}^{t_1} \dots \int_{\Lambda_n}^{t_{n-1}} e^{\lambda_1 t + (\lambda_2 - \lambda_1)t_1 + \dots + (\lambda_n - \lambda_{n-1})t_{n-1} - \lambda_n t_n}$$

$$\times \left[|f(y(t_n+\tau)) - f(y(t_n))| + |p(t_n+\tau) - p(t_n)| \right] dt_n \dots dt_1 \Bigg|$$

$$\leq \Bigg| \int_{\Lambda_{k+1}}^{t} \int_{\Lambda_{k+2}}^{t_{k+1}} \dots \int_{\Lambda_n}^{t_{n-1}} e^{\lambda_{k+1}t + (\lambda_{k+2}-\lambda_{k+1})t_{k+1} + \dots + (\lambda_n - \lambda_{n-1})t_{n-1} - \lambda_n t_n}$$

$$\times \left[\, L|y(t_n+\tau)-y(t_n)| + |p(t_n+\tau)-p(t_n)|\,\right] dt_n \dots dt_{k+1} \Bigg|$$

$$+ \sum_{j=1}^{k} \Bigg| \lambda_j \int\limits_{\Lambda_j}^{t} \int\limits_{\Lambda_{k+1}}^{t_j} \dots \int\limits_{\Lambda_n}^{t_{n-1}} e^{\lambda_1 t + (\lambda_2-\lambda_1)t_1 + \dots + (\lambda_n - \lambda_{n-1})t_{n-1} - \lambda_n t_n}$$

$$\times \left[\, L|y(t_n+\tau)-y(t_n)| + |p(t_n+\tau)-p(t_n)|\,\right] dt_n \dots dt_j \Bigg|$$

$$+ \sum_{\substack{i,j=1\\ i<j}}^{k} \Bigg| \lambda_i \lambda_j \int\limits_{\Lambda_i}^{t} \int\limits_{\Lambda_j}^{t_i} \int\limits_{\Lambda_{k+1}}^{t_j} \dots \int\limits_{\Lambda_n}^{t_{n-1}} e^{\lambda_i t + (\lambda_j-\lambda_i)t_i + \dots + (\lambda_n - \lambda_{n-1})t_{n-1} - \lambda_n t_n}$$

$$\times \left[\, L|y(t_n+\tau)-y(t_n)| + |p(t_n+\tau)-p(t_n)|\,\right] dt_n \dots dt_j \, dt_i \Bigg| + \dots$$

$$+ \Bigg| \left(\prod_{j=1}^{k} \lambda_j \right) \int\limits_{\Lambda_1}^{t} \int\limits_{\Lambda_2}^{t_1} \dots \int\limits_{\Lambda_n}^{t_{n-1}} e^{\lambda_1 t + (\lambda_2-\lambda_1)t_1 + \dots + (\lambda_n - \lambda_{n-1})t_{n-1} - \lambda_n t_n}$$

$$\times \left[\, L|y(t_n+\tau)-y(t_n)| + |p(t_n+\tau)-p(t_n)|\,\right] dt_n \dots dt_1 \Bigg|$$

$$= \frac{1}{\left|\prod\limits_{i=k+1}^{n}(-\lambda_i)\right|} \int\limits_0^1 \int\limits_0^1 \dots \int\limits_0^1 L \Bigg| y\Big(-\sum_{j=k+1}^{n} \frac{\ln s_j}{\lambda_j} + t + \tau\Big) - y\Big(-\sum_{j=k+1}^{n} \frac{\ln s_j}{\lambda_j} + t\Big) \Bigg|$$

$$+ \Bigg| p\Big(-\sum_{j=k+1}^{n} \frac{\ln s_j}{\lambda_j} + t + \tau\Big) - p\Big(-\sum_{j=k+1}^{n} \frac{\ln s_j}{\lambda_j} + t\Big) \Bigg| ds_n \dots ds_{k+2} \, ds_{k+1}$$

$$+ \frac{1}{\left|\prod\limits_{i=k+1}^{n}(-\lambda_i)\right|} \sum_{i=1}^{k} \int\limits_0^1 \int\limits_0^1 \dots \int\limits_0^1 L \Bigg| y\Big(-\frac{\ln s_i}{\lambda_i} - \sum_{j=k+1}^{n} \frac{\ln s_j}{\lambda_j} + t + \tau\Big)$$

$$- y\Big(-\frac{\ln s_i}{\lambda_i} - \sum_{j=k+1}^{n} \frac{\ln s_j}{\lambda_j} + t\Big) \Bigg|$$

$$+ \Bigg| p\Big(-\frac{\ln s_i}{\lambda_i} - \sum_{j=k+1}^{n} \frac{\ln s_j}{\lambda_j} + t + \tau\Big) - p\Big(-\frac{\ln s_i}{\lambda_i} - \sum_{j=k+1}^{n} \frac{\ln s_j}{\lambda_j} + t\Big) \Bigg| ds_n \dots ds_{k+1} \, ds_i$$

$$+ \dots + \frac{1}{\left|\prod\limits_{i=k+1}^{n}(-\lambda_i)\right|} \int\limits_0^1 \int\limits_0^1 \dots \int\limits_0^1 L \Bigg| y\Big(-\sum_{j=1}^{n} \frac{\ln s_j}{\lambda_j} + t + \tau\Big) - y\Big(-\sum_{j=1}^{n} \frac{\ln s_j}{\lambda_j} + t\Big) \Bigg|$$

$$+\left|p(-\sum_{j=1}^{n}\frac{\ln s_j}{\lambda_j}+t+\tau)-p(-\sum_{j=1}^{n}\frac{\ln s_j}{\lambda_j}+t)\right|ds_n\dots ds_2\,ds_1.$$

Thus, for arbitrary $a<b$, the following inequality holds:

$$\int_a^b |y^{(k)}(t+\tau)-y^{(k)}(t)|\,dt \le$$

$$\le \frac{1}{\prod_{i=k+1}^{n}|\lambda_i|}\int_a^b\int_0^1\dots\int_0^1 L\left|y(-\sum_{j=k+1}^{n}\frac{\ln s_j}{\lambda_j}+t+\tau)-y(-\sum_{j=k+1}^{n}\frac{\ln s_j}{\lambda_j}+t)\right|$$

$$+\left|p(-\sum_{j=k+1}^{n}\frac{\ln s_j}{\lambda_j}+t+\tau)-p(-\sum_{j=k+1}^{n}\frac{\ln s_j}{\lambda_j}+t)\right|ds_n\dots ds_{k+1}\,dt$$

$$+\frac{1}{\prod_{i=k+1}^{n}|\lambda_i|}\sum_{i=1}^{k}\int_a^b\int_0^1\int_0^1\dots\int_0^1 L\left|y(-\frac{\ln s_i}{\lambda_i}-\sum_{j=k+1}^{n}\frac{\ln s_j}{\lambda_j}+t+\tau)\right.$$

$$\left.-y(-\frac{\ln s_i}{\lambda_i}-\sum_{j=k+1}^{n}\frac{\ln s_j}{\lambda_j}+t)\right|$$

$$+\left|p(-\frac{\ln s_i}{\lambda_i}-\sum_{j=k+1}^{n}\frac{\ln s_j}{\lambda_j}+t+\tau)-p(-\frac{\ln s_i}{\lambda_i}-\sum_{j=k+1}^{n}\frac{\ln s_j}{\lambda_j}+t)\right|ds_n\dots ds_{k+1}\,ds_i\,dt$$

$$+\dots+\frac{1}{\prod_{i=k+1}^{n}|\lambda_i|}\int_a^b\int_0^1\dots\int_0^1 L\left|y(-\sum_{j=1}^{n}\frac{\ln s_j}{\lambda_j}+t+\tau)-y(-\sum_{j=1}^{n}\frac{\ln s_j}{\lambda_j}+t)\right|$$

$$+\left|p(-\sum_{j=1}^{n}\frac{\ln s_j}{\lambda_j}+t+\tau)-p(-\sum_{j=1}^{n}\frac{\ln s_j}{\lambda_j}+t)\right|ds_n\dots ds_1\,dt$$

$$=\frac{1}{\prod_{i=k+1}^{n}|\lambda_i|}\int_0^1\dots\int_0^1\int_a^b L\left|y(-\sum_{j=k+1}^{n}\frac{\ln s_j}{\lambda_j}+t+\tau)-y(-\sum_{j=k+1}^{n}\frac{\ln s_j}{\lambda_j}+t)\right|$$

$$+\left|p(-\sum_{j=k+1}^{n}\frac{\ln s_j}{\lambda_j}+t+\tau)-p(-\sum_{j=k+1}^{n}\frac{\ln s_j}{\lambda_j}+t)\right|dt\,ds_n\dots ds_{k+1}$$

$$+\frac{1}{\prod_{i=k+1}^{n}|\lambda_i|}\sum_{i=1}^{k}\int_0^1\int_0^1\dots\int_0^1\int_a^b L\left|y(-\frac{\ln s_i}{\lambda_i}-\sum_{j=k+1}^{n}\frac{\ln s_j}{\lambda_j}+t+\tau)\right.$$

$$- y(-\frac{\ln s_i}{\lambda_i} - \sum_{j=k+1}^{n} \frac{\ln s_j}{\lambda_j} + t)\Big|$$

$$+ \Big|p(-\frac{\ln s_i}{\lambda_i} - \sum_{j=k+1}^{n} \frac{\ln s_j}{\lambda_j} + t + \tau) - p(-\frac{\ln s_i}{\lambda_i} - \sum_{j=k+1}^{n} \frac{\ln s_j}{\lambda_j} + t)\Big|$$

$$\times dt\, ds_n \ldots ds_{k+1}\, ds_i$$

$$+ \ldots + \frac{1}{\prod\limits_{i=k+1}^{n} |\lambda_i|} \int_0^1 \ldots \int_0^1 \int_a^b L\Big|y(-\sum_{j=1}^{n} \frac{\ln s_j}{\lambda_j} + t + \tau) - y(-\sum_{j=1}^{n} \frac{\ln s_j}{\lambda_j} + t)\Big|$$

$$+ \Big|p(-\sum_{j=1}^{n} \frac{\ln s_j}{\lambda_j} + t + \tau) - p(-\sum_{j=1}^{n} \frac{\ln s_j}{\lambda_j} + t)\Big| dt\, ds_n \ldots ds_1.$$

Now, the $\bar{G}$-almost-periodicity of derivatives $y^{(k)}(\cdot)$ will be verified for single cases separately. Applying (iii) and employing to the correspondence between the ε-almost-period of $p(\cdot)$ and the $\frac{\varepsilon}{|a_n|-L}$-almost-period of solution $y(\cdot)$ (in the given pseudo-metric), one obtains e.g. in the Stepanov case (taking $a := u$, $b := u + 1$):

$$\sup_{u \in \mathbb{R}} \int_u^{u+1} |y^{(k)}(t + \tau) - y^{(k)}(t)|\, dt \le$$

$$\le \sup_{u \in \mathbb{R}} \frac{1}{\prod\limits_{i=k+1}^{n} |\lambda_i|} \Bigg(\int_0^1 \ldots \int_0^1 \int_u^{u+1} L\Big|y(-\sum_{j=k+1}^{n} \frac{\ln s_j}{\lambda_j} + t + \tau) - y(-\sum_{j=k+1}^{n} \frac{\ln s_j}{\lambda_j} + t)\Big|$$

$$+ \Big|p(-\sum_{j=k+1}^{n} \frac{\ln s_j}{\lambda_j} + t + \tau) - p(-\sum_{j=k+1}^{n} \frac{\ln s_j}{\lambda_j} + t)\Big|\, dt\, ds_n \ldots ds_{k+1}$$

$$+ \sum_{i=1}^{k} \int_0^1 \int_0^1 \ldots \int_0^1 \int_u^{u+1} L\Big|y(-\frac{\ln s_i}{\lambda_i} - \sum_{j=k+1}^{n} \frac{\ln s_j}{\lambda_j} + t + \tau) - y(-\frac{\ln s_i}{\lambda_i} - \sum_{j=k+1}^{n} \frac{\ln s_j}{\lambda_j} + t)\Big|$$

$$+ \Big|p(-\frac{\ln s_i}{\lambda_i} - \sum_{j=k+1}^{n} \frac{\ln s_j}{\lambda_j} + t + \tau) - p(-\frac{\ln s_i}{\lambda_i} - \sum_{j=k+1}^{n} \frac{\ln s_j}{\lambda_j} + t)\Big|$$

$$\times dt\, ds_n \ldots ds_{k+1}\, ds_i$$

$$+ \ldots + \int_0^1 \ldots \int_0^1 \int_u^{u+1} L\Big|y(-\sum_{j=1}^{n} \frac{\ln s_j}{\lambda_j} + t + \tau) - y(-\sum_{j=1}^{n} \frac{\ln s_j}{\lambda_j} + t)\Big|$$

$$+ \Big|p(-\sum_{j=1}^{n} \frac{\ln s_j}{\lambda_j} + t + \tau) - p(-\sum_{j=1}^{n} \frac{\ln s_j}{\lambda_j} + t)\Big|\, dt\, ds_n \ldots ds_1 \Bigg)$$

$$< \frac{1}{\prod\limits_{i=k+1}^{n} |\lambda_i|} \cdot \left\{ \frac{\varepsilon L}{|a_n| - L} + \varepsilon + \binom{k}{1} \left[\frac{\varepsilon L}{|a_n| - L} + \varepsilon \right] + \binom{k}{2} \left[\frac{\varepsilon L}{|a_n| - L} + \varepsilon \right] \right.$$

$$\left. + \ldots + \binom{k}{k} \left[\frac{\varepsilon L}{|a_n| - L} + \varepsilon \right] \right\} = \frac{|a_n|\, \varepsilon}{(|a_n| - L) \cdot \prod\limits_{i=k+1}^{n} |\lambda_i|} \sum_{j=0}^{k} \binom{k}{j}$$

$$= \frac{2^k |a_n|\, \varepsilon}{(|a_n| - L) \prod\limits_{i=k+1}^{n} |\lambda_i|}.$$

Following the similar way as above, we can prove quite analogously the equi-W-almost-periodicity of derivatives (taking $a := u$, $b := u + l$).

To verify the Weyl-almost-periodicity or the Besicovitch-almost-periodicity of derivatives $y^{(k)}(\cdot)$, we use the above integral estimate. Integrands contain the Weyl-a.p. or Besicovitch-a.p. function $p(\cdot)$ and entirely bounded, W-a.p. or B-a.p. solution $y(\cdot)$, respectively. Thus, we can verify by the similar manner as above the Weyl-almost-periodicity (putting $a := u$, $b := u + l$) as well as the Besicovitch-almost-periodicity ($a := -T$, $b = T$) of derivatives.

After all, to ε-almost-period of function $p(\cdot)$, there corresponds the $\frac{2^k |\lambda_1 \ldots \lambda_k|}{|a_n| - L}\varepsilon$-almost-period of k-th derivative ($k = 1, \ldots, n-1$) of solution $y(\cdot)$, in the given (pseudo-)metric, provided $L < |a_n|$. □

4 Existence of a.p. solutions: shifted polynomials approach

It is not very convenient that the almost-periods of the derivatives of a $\bar{G}$-a.p. solution depended on the roots of the characteristic polynomial (2). The shifted polynomials approach will allow us to avoid this handicap.

Theorem 2 *Let the above conditions (i)–(iv) be satisfied. Assume, furthermore, that $a_j \neq 0$, for $j = 1, \ldots, n-1$, and that all shifted polynomials $\lambda^{n-p} + \sum_{j=1}^{n-p} a_j \lambda^{n-p-j}$, $p = 0, \ldots, n-1$, have real nonzero roots. If $|a_n| > L$, then equation (1) admits a unique bounded $\bar{G}$-a.p. solution with $\bar{G}$-a.p. derivatives, up to the $(n-1)$-th order.*

Moreover, the ε-almost-period of $p(\cdot)$ implies the $\frac{2^k |a_n|}{|a_{n-k}|(|a_n| - L)}\, \varepsilon$-almost-period of the k-th derivative of the solution in the $\bar{G}$-(pseudo-)metric, for $k = 0, \ldots, n-1$.

Proof The existence of a unique bounded solution $y(\cdot)$ of (1) follows from Lemma 1. Its $\bar{G}$-almost-periodicity can be proved exactly in the same way as in the proof of Theorem 1. So, it remains to prove the $\bar{G}$-almost-periodicity of derivatives $y^{(k)}(\cdot)$. Putting $y(t)$ into f and substituting $\phi = y'$, one can write (1) in the form

$$\phi^{(n-1)} + \sum_{j=1}^{n-1} a_j \phi^{(n-j-1)} = f(y(t)) - a_n y(t) + p(t),$$

with exactly one bounded solution (again, according to Lemma 1). Applying the same procedure as at the beginning of the proof of Theorem 1, we can write the following inequality:

$$|\phi(t+\tau) - \phi(t)| \le$$

$$\le \frac{1}{|\widehat{\lambda}_{n-1}| \dots |\widehat{\lambda}_1|} \int_0^1 \dots \int_0^1 (L + |a_n|) \Big| y(-\sum_{j=1}^{n-1} \frac{\ln s_j}{\widehat{\lambda}_j} + t + \tau) - y(-\sum_{j=1}^{n-1} \frac{\ln s_j}{\widehat{\lambda}_j} + t) \Big|$$

$$+ \Big| p(-\sum_{j=1}^{n-1} \frac{\ln s_j}{\widehat{\lambda}_j} + t + \tau) - p(-\sum_{j=1}^{n-1} \frac{\ln s_j}{\widehat{\lambda}_j} + t) \Big| \, ds_{n-1} \dots ds_1,$$

where $\widehat{\lambda}_j \in \mathbb{R}$, $j = 1, \dots, n-1$, are nonzero roots of the corresponding characteristic polynomial $\lambda^{n-1} + \sum_{j=1}^{n-1} a_j \lambda^{n-1-j}$. Thus, for arbitrary $a < b$, the following estimate holds

$$\int_a^b |y'(t+\tau) - y'(t)| \, dt = \int_a^b |\phi(t+\tau) - \phi(t)| \, dt \le$$

$$\le \frac{L + |a_n|}{|\widehat{\lambda}_{n-1}| \dots |\widehat{\lambda}_1|} \int_a^b \int_0^1 \dots \int_0^1 \Big| y(-\sum_{j=1}^{n-1} \frac{\ln s_j}{\widehat{\lambda}_j} + t + \tau)$$

$$- y(-\sum_{j=1}^{n-1} \frac{\ln s_j}{\widehat{\lambda}_j} + t) \Big| \, ds_{n-1} \dots ds_1 \, dt$$

$$+ \frac{1}{|\widehat{\lambda}_{n-1}| \dots |\widehat{\lambda}_1|} \int_a^b \int_0^1 \dots \int_0^1 \Big| p(-\sum_{j=1}^{n-1} \frac{\ln s_j}{\widehat{\lambda}_j} + t + \tau)$$

$$- p(-\sum_{j=1}^{n-1} \frac{\ln s_j}{\widehat{\lambda}_j} + t) \Big| \, ds_{n-1} \dots ds_1 \, dt$$

$$= \frac{L + |a_n|}{|a_{n-1}|} \int_a^b \int_0^1 \dots \int_0^1 \Big| y(-\sum_{j=1}^{n-1} \frac{\ln s_j}{\widehat{\lambda}_j} + t + \tau) - y(-\sum_{j=1}^{n-1} \frac{\ln s_j}{\widehat{\lambda}_j} + t) \Big| \, ds_{n-1} \dots ds_1 \, dt$$

$$+ \frac{1}{|a_{n-1}|} \int_a^b \int_0^1 \dots \int_0^1 \Big| p(-\sum_{j=1}^{n-1} \frac{\ln s_j}{\widehat{\lambda}_j} + t + \tau) - p(-\sum_{j=1}^{n-1} \frac{\ln s_j}{\widehat{\lambda}_j} + t) \Big| \, ds_{n-1} \dots ds_1 \, dt.$$

To prove the $\bar{G}$-almost-periodicity of $y'(\cdot)$, denote by τ the ε-almost period of p and, subsequently, the $\frac{\varepsilon}{|a_n|-L}$-almost period of y (in the $\bar{G}$ (pseudo-)metric).

Concretely, for the S-almost-periodicity of $y'(\cdot)$, we apply the preceding inequality with $a = u$, $b = u+1$ and the fact that τ is the Stepanov ε-almost period of p as well as the Stepanov $\frac{\varepsilon}{|a_n|-L}$-almost period of y. Therefore,

$$\sup_{u\in\mathbb{R}} \int_u^{u+1} |y'(t+\tau) - y'(t)|\, dt < \frac{L+|a_n|}{|a_{n-1}|} \sup_{u\in\mathbb{R}} \int_u^{u+1}\int_0^1 \dots \int_0^1 \Big| y(-\sum_{j=1}^{n-1} \frac{\ln s_j}{\widehat{\lambda}_j} + t + \tau)$$

$$- y(-\sum_{j=1}^{n-1} \frac{\ln s_j}{\widehat{\lambda}_j} + t)\Big|\, ds_{n-1}\dots ds_1\, dt + \frac{\varepsilon}{|a_{n-1}|}$$

$$< \frac{L+|a_n|}{|a_{n-1}|} \cdot \frac{\varepsilon}{|a_n|-L} + \frac{\varepsilon}{|a_{n-1}|} = \frac{2|a_n|}{|a_{n-1}|\,(|a_n|-L)}\,\varepsilon.$$

Repeating the procedure with the equi-Weyl pseudo-metric, we obtain for $a = u$, $b = u+l$, where $l \geq l_0$, and for the equi-Weyl ε-almost period of p denoted by τ (which is the $\frac{\varepsilon}{|a_n|-L}$-almost period of solution y) that

$$\sup_{u\in\mathbb{R}} \frac{1}{l} \int_u^{u+l} |y'(t+\tau) - y'(t)|\, dt <$$

$$< \frac{L+|a_n|}{|a_{n-1}|} \sup_{u\in\mathbb{R}} \frac{1}{l} \int_u^{u+l}\int_0^1 \dots \int_0^1 \Big| y(-\sum_{j=1}^{n-1} \frac{\ln s_j}{\widehat{\lambda}_j} + t + \tau)$$

$$- y(-\sum_{j=1}^{n-1} \frac{\ln s_j}{\widehat{\lambda}_j} + t)\Big|\, ds_{n-1}\dots ds_1\, dt + \frac{\varepsilon}{|a_{n-1}|}$$

$$< \frac{L+|a_n|}{|a_{n-1}|} \cdot \frac{\varepsilon}{|a_n|-L} + \frac{\varepsilon}{|a_{n-1}|} = \frac{2|a_n|}{|a_{n-1}|\,(|a_n|-L)}\,\varepsilon.$$

This inequality holds for $\forall l \geq l_0$, where l_0 is connected with p.

The W-almost-periodicity of $y'(\cdot)$ will be proved in the same way. Denoting by τ the Weyl ε-almost period of p, one can derive:

$$\lim_{l\to+\infty} \sup_{u\in\mathbb{R}} \frac{1}{l} \int_u^{u+l} |y'(t+\tau) - y'(t)|\, dt <$$

$$< \frac{L+|a_n|}{|a_{n-1}|} \lim_{l\to+\infty} \sup_{u\in\mathbb{R}} \frac{1}{l} \int_u^{u+l}\int_0^1 \dots \int_0^1 \Big| y(-\sum_{j=1}^{n-1} \frac{\ln s_j}{\widehat{\lambda}_j} + t + \tau)$$

$$- y(-\sum_{j=1}^{n-1} \frac{\ln s_j}{\widehat{\lambda}_j} + t)\Big|\, ds_{n-1}\dots ds_1\, dt + \frac{\varepsilon}{|a_{n-1}|}$$

$$< \frac{L+|a_n|}{|a_{n-1}|} \cdot \frac{\varepsilon}{|a_n|-L} + \frac{\varepsilon}{|a_{n-1}|} = \frac{2|a_n|}{|a_{n-1}|\,(|a_n|-L)}\,\varepsilon.$$

Finally, let us concentrate on the Besicovitch case. Thanks to the above integral estimate, we can verify the B-almost periodicity of the derivative $y'(\cdot)$:

$$\limsup_{T\to+\infty} \frac{1}{2T} \int_{-T}^{+T} |y'(t+\tau)-y'(t)|\,dt <$$

$$< \frac{L+|a_n|}{|a_{n-1}|} \limsup_{T\to+\infty} \frac{1}{2T} \int_{-T}^{+T}\int_0^1 \dots \int_0^1 \Big| y\Big(-\sum_{j=1}^{n-1} \frac{\ln s_j}{\widehat{\lambda}_j} + t + \tau\Big)$$

$$- y\Big(-\sum_{j=1}^{n-1} \frac{\ln s_j}{\widehat{\lambda}_j} + t\Big)\Big|\, ds_{n-1}\dots ds_1\,dt + \frac{\varepsilon}{|a_{n-1}|} < \frac{2|a_n|}{|a_{n-1}|\,(|a_n|-L)}\,\varepsilon.$$

Hence, to ε-almost-period of p, there corresponds the $\frac{2|a_n|}{|a_{n-1}|(|a_n|-L)}\,\varepsilon$-almost-period of y', in the $\bar{G}$-(pseudo-)metric.

Putting $\psi = \phi'$, we arrive at the equation

$$\psi^{(n-2)} + \sum_{j=1}^{n-2} a_j \psi^{(n-j-2)} = f(y(t)) - a_n y(t) - a_{n-1} y'(t) + p(t).$$

In view of Lemma 1, this equation has exactly one entirely bounded solution. Proceeding by the similar way as above and denoting the roots of the corresponding characteristic polynomial by $\widehat{\widehat{\lambda}}_j$, $j = 1,\dots,n-2$, one gets the estimate

$$|\psi(t+\tau)-\psi(t)| \le \frac{1}{|\widehat{\widehat{\lambda}}_{n-2}|\dots|\widehat{\widehat{\lambda}}_1|} \int_0^1 \dots \int_0^1 (L+|a_n|)\Big| y\Big(-\sum_{j=1}^{n-2} \frac{\ln s_j}{\widehat{\widehat{\lambda}}_j} + t + \tau\Big)$$

$$- y\Big(-\sum_{j=1}^{n-2} \frac{\ln s_j}{\widehat{\widehat{\lambda}}_j} + t\Big)\Big| + |a_{n-1}|\Big| y'\Big(-\sum_{j=1}^{n-2} \frac{\ln s_j}{\widehat{\widehat{\lambda}}_j} + t + \tau\Big) - y'\Big(-\sum_{j=1}^{n-2} \frac{\ln s_j}{\widehat{\widehat{\lambda}}_j} + t\Big)\Big|$$

$$+ \Big| p\Big(-\sum_{j=1}^{n-1} \frac{\ln s_j}{\widehat{\widehat{\lambda}}_j} + t + \tau\Big) - p\Big(-\sum_{j=1}^{n-1} \frac{\ln s_j}{\widehat{\widehat{\lambda}}_j} + t\Big)\Big|\, ds_{n-2}\dots ds_1,$$

which leads (for $a < b$) to

$$\int_a^b |y''(t+\tau)-y''(t)|\,dt = \int_a^b |\psi(t+\tau)-\psi(t)|\,dt$$

$$\leq \frac{L+|a_n|}{|a_{n-2}|}\int\limits_a^b\int\limits_0^1\dots\int\limits_0^1\Big|y(-\sum_{j=1}^{n-2}\frac{\ln s_j}{\widehat{\widehat{\lambda}}_j}+t+\tau)-y(-\sum_{j=1}^{n-2}\frac{\ln s_j}{\widehat{\widehat{\lambda}}_j}+t)\Big|$$
$$\times\, ds_{n-2}\dots ds_1\, dt$$
$$+\frac{|a_{n-1}|}{|a_{n-2}|}\int\limits_a^b\int\limits_0^1\dots\int\limits_0^1\Big|y'(-\sum_{j=1}^{n-2}\frac{\ln s_j}{\widehat{\widehat{\lambda}}_j}+t+\tau)-y'(-\sum_{j=1}^{n-2}\frac{\ln s_j}{\widehat{\widehat{\lambda}}_j}+t)\Big|\,ds_{n-2}\dots ds_1\,dt$$
$$+\frac{1}{|a_{n-2}|}\int\limits_a^b\int\limits_0^1\dots\int\limits_0^1\Big|p(-\sum_{j=1}^{n-2}\frac{\ln s_j}{\widehat{\widehat{\lambda}}_j}+t+\tau)-p(-\sum_{j=1}^{n-2}\frac{\ln s_j}{\widehat{\widehat{\lambda}}_j}+t)\Big|\,ds_{n-2}\dots ds_1\,dt.$$

Proceeding by the same way as in the case of $y'(\cdot)$, we can analyze all kinds of (pseudo-)metrics separately. The $\bar{G}$-almost-periodicity of $y''(\cdot)$ can be verified by means of the ε-almost-period of p (denoted by τ, as usual), which coincides with the $\frac{\varepsilon}{|a_n|-L}$-almost period of solution y and the $\frac{2|a_n|}{|a_{n-1}|(|a_n|-L)}$ ε-almost-period of y', in the $\bar{G}$-(pseudo-)metric. Repeating the procedure as above, we get that the mentioned almost-period τ coincides with the $\frac{4|a_n|}{|a_{n-2}|(|a_n|-L)}$ ε-almost-period of y'', in the $\bar{G}$-(pseudo-)metric.

By the same manner, we can verify the $\bar{G}$-almost-periodicity of higher-order derivatives $y^{(k)}$. The essential estimate takes now the form

$$\int\limits_a^b|y^{(k)}(t+\tau)-y^{(k)}(t)|\,dt\leq$$

$$\leq\frac{L+|a_n|}{\prod_{j=1}^{n-k}|\widetilde{\lambda}_j|}\int\limits_a^b\int\limits_0^1\dots\int\limits_0^1\Big|y(-\sum_{j=1}^{n-k}\frac{\ln s_j}{\widetilde{\lambda}_j}+t+\tau)-y(-\sum_{j=1}^{n-k}\frac{\ln s_j}{\widetilde{\lambda}_j}+t)\Big|\,ds_{n-k}\dots ds_1\,dt$$
$$+\sum_{l=1}^{k-1}\Bigg(\frac{|a_{n-l}|}{\prod_{j=1}^{n-k}|\widetilde{\lambda}_j|}\int\limits_a^b\int\limits_0^1\dots\int\limits_0^1\Big|y^{(l)}(-\sum_{j=1}^{n-k}\frac{\ln s_j}{\widetilde{\lambda}_j}+t+\tau)$$
$$-y^{(l)}(-\sum_{j=1}^{n-k}\frac{\ln s_j}{\widetilde{\lambda}_j}+t)\Big|\,ds_{n-k}\dots ds_1\,dt\Bigg)$$
$$+\frac{1}{\prod_{j=1}^{n-k}|\widetilde{\lambda}_j|}\int\limits_a^b\int\limits_0^1\dots\int\limits_0^1\Big|p(-\sum_{j=1}^{n-k}\frac{\ln s_j}{\widetilde{\lambda}_j}+t+\tau)-p(-\sum_{j=1}^{n-k}\frac{\ln s_j}{\widetilde{\lambda}_j}+t)\Big|$$
$$\times\, ds_{n-k}\dots ds_1\,dt,$$

for $a<b$, where $\widetilde{\lambda}_j\in\mathbb{R}$ denote the nonzero roots of the related shifted polynomial ($j=1,\dots,n-k$).

Studying all cases separately, we obtain the $\bar{G}$-almost-periodicity of $y^{(k)}$. Moreover, the relationship between $\bar{G}$-almost-periods of p and of derivatives $y^{(k)}$ can be described as follows: to ε-almost-period of p, there corresponds the $\frac{2^k|a_n|}{|a_{n-k}|(|a_n|-L)}\,\varepsilon$-almost-period of $y^{(k)}$, in the $\bar{G}$-(pseudo-)metric, for $k = 0,\ldots,n-1$, provided $L < |a_n|$, and $|a_j| \neq 0$, for $j = 0,\ldots,n$. This completes the proof. □

5 Existence of a.p. solutions: case of negative roots

Another way how to come to almost-periods of the derivatives not depending on the roots of the characteristic polynomial $\lambda^n + \sum_{j=1}^n a_j\lambda^{n-j}$ is to assume that all roots are negative. This implies that all coefficients a_j, $j = 1,\ldots,n$, must be positive.

Theorem 3 *Let the above conditions (i)–(iv) be satisfied. Assume additionally that all roots of the characteristic polynomial (2) are negative. If $L < a_n$, then there exists a unique bounded $\bar{G}$-a.p. solution $y(\cdot)$ of equation (1) with $\bar{G}$-a.p. derivatives, up to the $(n-1)$-th order.*

Moreover, the ε-almost-period of $p(\cdot)$ implies the $\frac{2^k a_k}{\binom{n}{k}(a_n-L)}\,\varepsilon$-almost-period of the k-th derivative $y^{(k)}(\cdot)$ of the solution $y(\cdot)$, in the $\bar{G}$-(pseudo-)metric, for $k = 0,\ldots,n-1$, where $a_0 := 1$.

Proof According to Lemma 1, equation (1) admits exactly one bounded solution. Its representation formula can be now written in the form:

$$y(t) = \int_{-\infty}^{t}\int_{-\infty}^{t_1}\cdots\int_{-\infty}^{t_{n-1}} e^{\lambda_1 t+(\lambda_2-\lambda_1)t_1+\ldots+(\lambda_n-\lambda_{n-1})t_{n-1}-\lambda_n t_n} \times [f(y(t_n)) + p(t_n)]\,dt_n\ldots dt_1.$$

Analogously as in the proof of Theorem 1, we can prove the $\bar{G}$-almost-periodicity of solution $y(\cdot)$.

Proceeding by the same way as in the proof of Theorem 1, we can check the $\bar{G}$-almost-periodicity of derivatives $y^{(k)}(\cdot)$, $k = 1,\ldots,n-1$. More precisely, we can specify that the ε-almost-period of $p(\cdot)$ implies the $\frac{2^k(-1)^k\lambda_1\ldots\lambda_k}{a_n-L}\,\varepsilon$-almost-period of k-th derivative of solution $y(\cdot)$, $k = 1,\ldots,n-1$, in the $\bar{G}$-sense. Due to the independence of the preceding term under the permutation of roots (see [AG, p. 326]), one has $\binom{n}{k}$ choices of $\lambda_{i_1},\ldots,\lambda_{i_k}$ for n roots of the characteristic polynomial (2). Let us sum up the following $\binom{n}{k}$ inequalities:

$$\sup_{u\in\mathbb{R}}\int_u^{u+1}|y^{(k)}(t+\tau) - y^{(k)}(t)|\,dt < \frac{2^k(-1)^k\,\lambda_{i_1}\ldots\lambda_{i_k}}{a_n - L}\,\varepsilon,$$

for the S-metric,

$$\sup_{u\in\mathbb{R}} \frac{1}{l} \int\limits_u^{u+l} |y^{(k)}(t+\tau) - y^{(k)}(t)|\, dt < \frac{2^k(-1)^k\, \lambda_{i_1} \dots \lambda_{i_k}}{a_n - L}\, \varepsilon,$$

in the equi-Weyl case (for all $l \geq l_0$, where l_0 is connected with p),

$$\lim_{l\to+\infty} \sup_{u\in\mathbb{R}} \frac{1}{l} \int\limits_u^{u+l} |y^{(k)}(t+\tau) - y^{(k)}(t)|\, dt < \frac{2^k(-1)^k\, \lambda_{i_1} \dots \lambda_{i_k}}{a_n - L}\, \varepsilon,$$

for the Weyl pseudo-metric, and

$$\limsup_{T\to+\infty} \frac{1}{2T} \int\limits_{-T}^{T} |y^{(k)}(t+\tau) - y^{(k)}(t)|\, dt < \frac{2^k(-1)^k\, \lambda_{i_1} \dots \lambda_{i_k}}{a_n - L}\, \varepsilon,$$

in the Besicovitch case.

Divide these sums by $\binom{n}{k}$. Now, application of the Vieta formula

$$\sum_{\substack{i_1,\dots,i_k=1\\ i_1<\dots<i_k}}^{n} (-1)^k \prod_{j=1}^{k} \lambda_{i_j} = a_k$$

leads to the desired simplification: every ε-almost-period of $p(\cdot)$ implies the $\frac{2^k a_k}{\binom{n}{k}(a_n-L)}\, \varepsilon$-almost-period of $y^{(k)}(\cdot)$, in the $\bar{G}$-sense. □

6 Concluding remarks

First of all, one can readily check that all main theorems remain valid if, instead of the boundedness of f, only the existence of a positive constant $D_0 > 0$ is assumed such that (cf. Remark 2)

$$\max_{|y|\leq D_0/|a_n|} |f(y)| + \operatorname{supess}_{t\in\mathbb{R}} |p(t)| \leq D_0.$$

The same is true for the Lipschitzianity of f: it is enough that

$$|f(x) - f(y)| \leq L\, |x - y|$$

holds, with $0 < L < |a_n|$, only for $|x| \leq \frac{D_0}{|a_n|}$, $|y| \leq \frac{D_0}{|a_n|}$.

Therefore, considering the pendulum-type equation

$$y'' + ay' + b \sin y = p(t), \tag{3}$$

where a, b are nonzero constants such that $a^2 \geq 4|b|$ and $p \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R},\mathbb{R})$ is $\overline{G}$-a.p., and following the arguments in [A1] (cf. [AG, pp. 556–557]), we can easily deduce that eguation (3) *admits at least two $\overline{G}$-a.p. solutions $y_1(\cdot)$ and $y_2(\cdot)$ with $\overline{G}$-a.p. derivatives such that*

$$\sup_{t\in(-\infty,\infty)} |y_1(t)| < \frac{\pi}{2} \qquad \text{and} \qquad \sup_{t\in(-\infty,\infty)} |y_2(t)-\pi| < \frac{\pi}{2},$$

provided only

$$\operatorname*{supess}_{t\in(-\infty,\infty)} |p(t)| < |b|.$$

Furthermore, since multivalued Lipschitz-continuous function with nonempty, convex and compact values $\varphi : \mathbb{R} \to 2^{\mathbb{R}} \setminus \{\emptyset\}$, i.e.

$$\mathrm{d}_H\,(\varphi(x), \varphi(y)) \leq L\,|x-y|,$$

where d_H stands for the Hausdorff metric and $L \in \mathbb{R}$ is a constant, possesses a single-valued Lipschitz continuous selection $f \subset \varphi$ with constant L_0 such that $L_0 := L(12\sqrt{3}/5+1)$ (see e.g. [HP, pp. 101–103]), and since Stepanov or equi-Weyl a.p. multivalued function with nonempty, convex and compact values $P : \mathbb{R} \to 2^{\mathbb{R}} \setminus \{\emptyset\}$ possesses a single-valued Stepanov or equi-Weyl a.p. selection $p \subset P$, respectively (see [D1], [D2], [DS] resp. [D3]), the existence parts (without uniqueness) of all main theorems can be extended to the differential inclusions

$$y^{(n)} + \sum_{j=1}^{n} a_j y^{(n-j)} \in \varphi(y) + P(t),$$

provided a positive constant $D_0 > 0$ exists such that

$$\max_{|y|\leq D_0/|a_n|} |\varphi(y)| + \operatorname*{supess}_{t\in\mathbb{R}} |P(t)| \leq D_0,$$

φ is Lipschitz-continuous, for $|y| \leq \frac{D_0}{|a_n|}$, with a constant L such that

$$L < |a_n|/(12\sqrt{3}/5+1),$$

and P is either Stepanov or equi-Weyl almost-periodic in a multivalued sense (for the related definitions and more details, see e.g. [AG, Chapter III.10 and Appendix A.1]).

Finally, all $\overline{G}$-a.p. solutions $y(\cdot)$ and their derivatives, up to the order $(n-1)$, are in fact (see Remark 3) $\overline{G}$-normal (a.p. in the sense of Bochner), i.e. the families $\{y^{(k)}(t+h) \mid h \in \mathbb{R}\}$, $h = 0, 1, \ldots, n-1$, are $\overline{G}$-precompact, because these solutions and their derivatives are bounded and uniformly continuous; for more details, see [AG, Chapter III.10] and [ABG]. Stepanov a.p. solutions are even uniformly almost-periodic.

Some further remarks are in order.

Remark 5 Observe the similarity of the estimates for ε-almost-periods with those for bounded solutions and their derivatives in Remark 2.

Remark 6 Analogous theorems can be obtained when only assuming that the real parts of the roots of the characteristic polynomial (2) are nonzero. On the other hand, the explicit inequality $L < |a_n|$ would be replaced by a rather implicit condition $L < |\alpha_1 \ldots \alpha_n|$, where $\alpha_j = \operatorname{Re}\lambda_j$, $j = 1, \ldots, n$, denote the real parts of the roots λ_j of (2). Moreover, the related ε-almost-periods of a.p. solutions and their derivatives would depend on α_j, $j = 1, \ldots, n$.

Remark 7 Similar theorems can be also deduced for a more general equation than (1), namely

$$y^{(n)} + \sum_{j=1}^{n} a_j y^{(n-j)} = \sum_{j=1}^{n} f_j \left(y^{(n-j)} \right) + p(t),$$

or inclusion (without uniqueness)

$$y^{(n)} + \sum_{j=1}^{n} a_j y^{(n-j)} \in \sum_{j=1}^{n} \varphi_j \left(y^{(n-j)} \right) + P(t),$$

but the related calculations would be rather cumbersome. At least in the case of uniformly a.p. solutions, this will be treated by ourselves elsewhere.

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$2-(n^2,2n,2n-1)$ Designs Obtained from Affine Planes*

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Abstract

The simple incidence structure $\mathcal{D}(\mathcal{A},2)$ formed by points and unordered pairs of distinct parallel lines of a finite affine plane $\mathcal{A}=(\mathcal{P},\mathcal{L})$ of order $n>2$ is a $2-(n^2,2n,2n-1)$ design. If $n=3$, $\mathcal{D}(\mathcal{A},2)$ is the complementary design of $\mathcal{A}$. If $n=4$, $\mathcal{D}(\mathcal{A},2)$ is isomorphic to the geometric design $AG_3(4,2)$ (see [2; Theorem 1.2]). In this paper we give necessary and sufficient conditions for a $2-(n^2,2n,2n-1)$ design to be of the form $\mathcal{D}(\mathcal{A},2)$ for some finite affine plane $\mathcal{A}$ of order $n>4$. As a consequence we obtain a characterization of small designs $\mathcal{D}(\mathcal{A},2)$.

Key words: $2-(n^2,2n,2n-1)$ designs; incidence structure; affine planes.

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By a $2-(v,k,\lambda)$ *design* we mean a pair $\mathcal{D}=(\mathcal{P},\mathcal{B})$ where $\mathcal{P}$ is a set of v points and $\mathcal{B}$ is a collection of distinguished subsets of $\mathcal{P}$ called blocks such that each block contains k points and any two distinct points are contained in exactly λ common blocks[1]. Our main result is the following

Theorem 1 *Let n be an integer with $n>4$ and let $\mathcal{D}=(\mathcal{P},\mathcal{B})$ be a $2-(n^2,2n,2n-1)$ design. Then $\mathcal{D}$ is of the form $\mathcal{D}(\mathcal{A},2)$ if and only if the following two conditions are satisfied: (c_1) any three distinct points of $\mathcal{D}$*

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[1]For further definitions (and basic results) about 2-designs see [1].

are contained in exactly 3 *or* $n-1$ *common blocks; (c*$_2$*) if* $X_1, X_2, \ldots, X_{n-1}$ *are* $n-1$ *distinct blocks of* $\mathcal{D}$ *such that* $|X_1 \cap X_2 \cap \cdots \cap X_{n-1}| > 2$*, then* $X_1 \cap X_2 \cap \cdots \cap X_{n-1} = X_i \cap X_j$ *whenever* $i \neq j$*.*

Before proving the theorem we need some preliminary results about $2-(n^2, 2n, 2n-1)$ designs.

Lemma 1 *Suppose* $\mathcal{A} = (\mathcal{P}, \mathcal{L})$ *is a finite affine plane of order* $n > 4$ *and let* $\mathcal{D}(\mathcal{A}, 2)$ *be the system of points and unordered pairs of distinct parallel lines of* $\mathcal{A}$*. Then* $\mathcal{D}(\mathcal{A}, 2)$ *is a* $2-(n^2, 2n, 2n-1)$ *design satisfying the following properties:*

(1) any three distinct collinear points of $\mathcal{A}$ *are contained in exactly* $n-1$ *blocks of* $\mathcal{D}(\mathcal{A}, 2)$*;*

(2) any three distinct non-collinear points of $\mathcal{A}$ *are joined by precisely 3 blocks of* $\mathcal{D}(\mathcal{A}, 2)$*;*

(3) if $X_1, X_2, \ldots, X_{n-1}$ *are* $n-1$ *distinct blocks of* $\mathcal{D}(\mathcal{A}, 2)$ *such that* $|X_1 \cap X_2 \cap \cdots \cap X_{n-1}| > 2$*, then* $X_1 \cap X_2 \cap \cdots \cap X_{n-1} = X_i \cap X_j$ *whenever* $i \neq j$*.*

Proof This follows directly from the definition of $\mathcal{D}(\mathcal{A}, 2)$. □

Lemma 2 *Let* n *be an integer greater than* 4 *and let* $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ *be a* $2-(n^2, 2n, 2n-1)$ *design any three distinct points of which are contained in exactly 3 or* $n-1$ *blocks. Then for any choice of two distinct points* x, y *in* $\mathcal{D}$ *there are precisely* $n-2$ *points* $z \in \mathcal{P} \setminus \{x, y\}$ *with the property that* x, y, z *are joined by* $n-1$ *distinct blocks of* $\mathcal{D}$*.*

Proof Let x, y be any two distinct points of $\mathcal{D}$ and denote by c the number of points $z \in \mathcal{P} \setminus \{x, y\}$ with the property that x, y, z are joined by $n-1$ blocks of $\mathcal{D}$. Then $0 \leq c \leq n^2 - 2$ and $n^2 - 2 - c$ is the number of points $w \in \mathcal{P} \setminus \{x, y\}$ with the property that x, y, w are joined by exactly 3 blocks of $\mathcal{D}$. Thus, counting the point block pairs (p, C) with $x \neq p \neq y$ and $\{x, y, p\} \subset C$, we find $3(n^2 - 2 - c) + (n-1)c = (2n-2)(2n-1)$ which can be written as $(n-4)c = (n-4)(n-2)$. Hence, since $n - 4 \neq 0$, $c = n-2$ and the lemma is proved. □

Lemma 3 *Let* n *be an integer with* $n > 4$ *and let* $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ *be a* $2-(n^2, 2n, 2n-1)$ *design. If* $X_1, X_2, \ldots, X_{n-1}$ *are* $n-1$ *distinct blocks of* $\mathcal{D}$ *such that* $X_1 \cap X_2 \cap \cdots \cap X_{n-1} = X_i \cap X_j$ *whenever* $i \neq j$*, then* $|X_1 \cap X_2 \cap \cdots \cap X_{n-1}| \geq n$ *with equality if and only if* $X_1 \cup X_2 \cup \ldots X_{n-1} = \mathcal{P}$*.*

Proof Write $X_1 \cup X_2 \cup \cdots \cup X_{n-1} = l \cup (X_1 \setminus l) \cup (X_2 \setminus l) \cup \cdots \cup (X_{n-1} \setminus l)$, where $l = X_1 \cap X_2 \cap \cdots \cap X_{n-1}$. Then $|X_1 \cup X_2 \cup \cdots \cup X_{n-1}| = a + (n-1)(2n-a) = n^2 + (n-2)(n-a)$ with $a = |l|$. Thus, since $\mathcal{D}$ has n^2 points, we obtain $n^2 \geq n^2 + (n-2)(n-a)$ which, since $n > 4$, gives $n \leq a$. Moreover $n = a$ is equivalent to ask $|X_1 \cup X_2 \cup \cdots \cup X_{n-1}| = n^2$, i.e. $X_1 \cup X_2 \cup \cdots \cup X_{n-1} = \mathcal{P}$, and the lemma is proved. □

Proof of Theorem 1 In view of Lemma 1, we have only to prove that $\mathcal{D} = \mathcal{D}(\mathcal{A}, 2)$ for some affine plane $\mathcal{A}$ (of order n), provided conditions (c_1) and (c_2) hold. Define $\mathcal{A} = (\mathcal{P}, \mathcal{L})$ by taking $\mathcal{P}$ as the set of points and the set $\mathcal{L} = \{l \subset P : |l| > 2, l = L_1 \cap L_2 \cap \cdots \cap L_{n-1}$ with $L_1, L_2, \ldots, L_{n-1}$ distinct blocks of $\mathcal{D}\}$ as the set of lines. By Lemma 2, $\mathcal{L}$ is non empty. Let $l \in \mathcal{L}$ and let $L_1, L_2, \ldots, L_{n-1}$ be the $n-1$ distinct blocks of $\mathcal{D}$ such that $l = L_1 \cap L_2 \cap \cdots \cap L_{n-1}$. Then condition (c_2) gives $l = L_i \cap L_j$ whenever $i \neq j$ so that, by Lemma 3, l contains at least n points. On the other hand, as any three distinct points of l are joined by the $n-1$ blocks L_i $(i = 1, 2, \ldots, n-1)$, it follows from Lemma 2 that l contains at most $2 + (n-2) = n$ points. Thus we must have $n \leq |l| \leq n$ and consequently $|l| = n$. Let x, y be any two distinct points of $\mathcal{D}$. By Lemma 2 we may choose a point $z \in \mathcal{P} \setminus \{x, y\}$ and $n-1$ distinct blocks $Z_1, Z_2, \ldots, Z_{n-1} \in \mathcal{B}$ such that $\{x, y, z\} \subseteq Z_1 \cap Z_2 \cap \cdots \cap Z_{n-1}$. Therefore $h = Z_1 \cap Z_2 \cap \ldots Z_{n-1}$ belongs to $\mathcal{L}$ and passes through both x and y. Assume that $\{x, y\} \subseteq k$ for some $k \in \mathcal{L}$ with $k \neq h$. Writing k as the intersection $k = W_1 \cap W_2 \cap \ldots W_{n-1}$ of $n-1$ distinct blocks $W_1, W_2, \ldots, W_{n-1} \in \mathcal{B}$ we obtain $\{x, y, p\} \subseteq Z_1 \cap Z_2 \cap \cdots \cap Z_{n-1}$ or $\{x, y, p\} \subseteq W_1 \cap W_2 \cap \cdots \cap W_{n-1}$ whenever $p \in h \cup k$ is a point such that $x \neq p \neq y$. Then from Lemma 2 we deduce $|h \cup k| \leq 2 + (n-2) = n$ which contradicts our assumption $k \neq h$ and shows that h is the unique element in $\mathcal{L}$ containing $\{x, y\}$. Thus each $l \in \mathcal{L}$ has n points and each pair of points is on exactly one common point set $m \in \mathcal{L}$: this is sufficient to conclude that $\mathcal{A} = (\mathcal{P}, \mathcal{L})$ is a finite affine plane of order n. Note that such a plane $\mathcal{A} = (\mathcal{P}, \mathcal{L})$ has the properties: (i) for any line $l \in \mathcal{L}$ and any point $x \in \mathcal{P}, x \notin l$, there is just one block of $\mathcal{D}$ containing both l and x; (ii) if a block $C \in \mathcal{B}$ contains a line $h \in \mathcal{L}$ and if $y \in C$ is a point not on h, then $C = h \cup k$ where $k \in \mathcal{L}$ is the only line of $\mathcal{A}$ through y not intersecting h. Property (i) follows from the fact that (by condition (c_2) and Lemma 3) the point set $\mathcal{P}$ can be written as disjoint union $P = l \cup (L_1 \setminus l) \cup (L_2 \setminus l) \cup \cdots \cup (L_{n-1} \setminus l)$, if $L_1, L_2, \ldots, L_{n-1}$ are the $n-1$ distinct blocks of $\mathcal{D}$ through the line $l \in \mathcal{L}$. To show (ii) we proceed as follows. Denote by k the line of $\mathcal{A}$ through y parallel to h. Let $z \in C \setminus h$ be a point distinct from y and denote by l the line of $\mathcal{A}$ joining y to z. We claim that $l = k$. In fact $l \neq h$ and $l = W_1 \cap W_2 \cap \cdots \cap W_{n-1}$ for suitable $n-1$ distinct blocks $W_1, W_2, \ldots, W_{n-1} \in \mathcal{B}$. Suppose there is a point $w \in h \cap l$. Then y, z, w are three distinct points belonging to l and, by condition (c_1), there is no block in $\mathcal{D}$ containing $\{y, z, w\}$, apart from the blocks W_i. But $h \subset C$ forces $w \in C$ and consequently $\{y, z, w\} \subset C$. Thus we have $C = W_i$ for some $i \in \{1, 2, \ldots, n-1\}$ so that $l \subset C$. Then $l \cup h \subseteq C$ and there is just one point $p \in C$ such that $p \notin l \cup h$, since $|C| = 2n = 1 + |l \cup h|$. As p belongs to $n+1$ lines of $\mathcal{A}$, we may choose a line $s \in \mathcal{L}$ through p such that $w \notin s$ and s meets both l and h. Since $C = \{p\} \cup l \cup h$, we have that s intersects C in exactly three points, namely $p, l \cap s$ and $h \cap s$. On the other hand, if $S_1, S_2, \ldots, S_{n-1}$ are the $n-1$ distinct blocks of $\mathcal{D}$ such that $s = S_1 \cap S_2 \cap \cdots \cap S_{n-1}$, we infer from condition (c_1) that $S_1, S_2, \ldots, S_{n-1}$ are the only blocks of $\mathcal{D}$ containing $p, l \cap s, h \cap s$. Since

$\{p, l \cap s, h \cap s\} \subset C$, we obtain $C = S_j$ for some $j \in \{1, 2, \ldots, n-1\}$ and hence $s \subset C$. Therefore $s = s \cap C$ consists of three points, a contradiction. Thus l and h do not intersect and l is the unique line of $\mathcal{A}$ through y not intersecting h, i.e. $l = k$. Therefore $z \in k$. As this is true for every point $z \in C \setminus h$ distinct from y and $|C \setminus h| = n = |k|$, we may conclude that $C \setminus h = k$. So $C = h \cup k$ and (ii) holds.

As any parallel class of the affine plane $\mathcal{A} = (\mathcal{P}, \mathcal{L})$ consists of n lines and $\mathcal{A}$ has $n+1$ parallel classes, we infer from (i) and (ii) that $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ contains exactly $(n+1)\frac{n(n-1)}{2}$ blocks X of the form $X = l \cup m$ with l, m distinct parallel lines of $\mathcal{A}$. But any $2-(n^2, 2n, 2n-1)$ design has precisely $b = (n+1)\frac{n(n-1)}{2}$ blocks. Then we must have

$$\mathcal{B} = \{X \subset \mathcal{P} : X = l \cup m \text{ with } l, m \text{ distinct parallel lines of } \mathcal{A}\}$$

and hence $\mathcal{D} = \mathcal{D}(\mathcal{A}, 2)$. The theorem is proved. □

Since up to isomorphism there is just one affine plane of order $5, 7$ or 8 we have the following characterization of small designs $\mathcal{D}(\mathcal{A}, 2)$.

Corollary 1 *Suppose n is one of the numbers $5, 7, 8$ and let $\mathcal{A}(n)$ be the desarguesian affine plane of order n. There exists up to isomorphisms exactly one $2-(n^2, 2n, 2n-1)$ design $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ satisfying conditions (c_1), (c_2) of Theorem 1, namely the 2-design $\mathcal{D}(\mathcal{A}(n), 2)$.*

We end our investigation with a few remarks

Remark 1 If $\mathcal{A} = (\mathcal{P}, \mathcal{L})$ is a finite affine plane of order $n > 4$, then $0, 4, n$ are the intersection numbers of the $2-(n^2, 2n, 2n-1)$ design $\mathcal{D}(\mathcal{A}, 2)$: i.e. $\{0, 4, n\} = \{|X \cap Y| : X, Y$ are two distinct blocks of $\mathcal{D}(\mathcal{A}, 2)\}$.

Remark 2 There is no plane of order $n = 6$, but there is an example of a $2-(36, 12, 11)$ design produced by H. Hanany [3], Table 5.23, p. 343. The $2-(25, 10, 9)$ design $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ exhibited by H. Hanany, loc. cit. Table 5.23, p. 334 is not of the form $\mathcal{D}(\mathcal{A}, 2)$: since $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ admits 8 as an intersection number (i.e. $|X \cap Y| = 8$ for suitable distinct blocks $X, Y \in \mathcal{B}$).

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Directoids with Sectionally Switching Involutions*

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Abstract

It is shown that every directoid equipped with sectionally switching mappings can be represented as a certain implication algebra. Moreover, if the directoid is also commutative, the corresponding implication algebra is defined by four simple identities.

Key words: Directoid; commutative directoid; semilattice; involution; implication algebra; sectionally switching mapping.

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The concept of directoid was introduced by J. Ježek and R. Quackenbush [4] in the sake to axiomatize algebraic structures defined on upward directed ordered sets. In certain sense, directoids generalize semilattices. For the reader convenience, we repeat definitions and basic properties of these concepts.

An ordered set $(A;\leq)$ is *upward directed* if $U(x,y)\neq\emptyset$ for every $x,y\in A$, where $U(x,y)=\{a\in A;\ x\leq a \text{ and } y\leq a\}$. Elements of $U(x,y)$ are referred to be common upper bounds of x,y. Of course, if $(A;\leq)$ has a greatest element then it is upward directed.

Let $(A;\leq)$ be an upward directed set and $\sqcup$ denots a binary operation on A. The pair $\mathcal{A}=(A;\sqcup)$ is called a *directoid* if

(i) $x\sqcup y\in U(x,y)$ for all $x,y\in A$;

(ii) if $x\leq y$ then $x\sqcup y=y$ and $y\sqcup x=y$.

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If, moreover, the operation $\sqcup$ is commutative, $\mathcal{A}$ is called a *commutative directoid.*

Example 1 Consider an ordered set $A = \{a, b, c, d, 1\}$ whose diagram is visualized in Fig. 1.

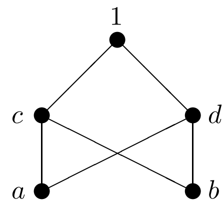

Fig. 1

Define $a \sqcup b = d$, $b \sqcup a = c$, $c \sqcup d = d \sqcup c = 1$ and for other couples $x, y \in A$ by the condition (ii). Then $\mathcal{A} = (A; \sqcup)$ is a directoid which is not commutative.

Of course, every $\vee$-semilattice is a commutative directoid. When we change in our Example 1 the definition of $\sqcup$ only in one instance, i.e. we put $b \sqcup a = d$, the resulting algebra is a commutative directoid which is not a semilattice.

The following axiomatization of directoids was involved in [4]:

Proposition 1 *A groupoid* $\mathcal{A} = (A; \sqcup)$ *is a directoid if and only if it satisfies the following identities*

(D1) $x \sqcup x = x$;

(D2) $(x \sqcup y) \sqcup x = x \sqcup y$;

(D3) $y \sqcup (x \sqcup y) = x \sqcup y$;

(D4) $x \sqcup ((x \sqcup y) \sqcup z) = (x \sqcup y) \sqcup z.$

Then a binary relation $\leq$ *defined on* A *by the rule*

$$x \leq y \text{ if and only if } x \sqcup y = y \qquad (R)$$

is an order and $x \sqcup y \in U(x, y)$ *for each* $x, y \in A$.

A groupoid $\mathcal{A} = (A; \sqcup)$ *is a commutative directoid if and only if it satisfies the identities* (D1), (D4) *and*

(D5) $x \sqcup y = y \sqcup x.$

Let us note that if a directoid $\mathcal{A} = (A; \sqcup)$ is associative, i.e. if it satisfies the identity $x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$ then it is also commutative and hence a semilattice.

Of course, every upward directed set $(A; \leq)$ can be converted into a (comutative) directoid whenever one assignes to a couple $x, y \in A$ an element $\lambda(x, y) \in U(x, y)$ such that for $x \leq y$ we pick up $\lambda(x, y) = \lambda(y, x) = y$. Then for $x \sqcup y = \lambda(x, y)$, $(A; \sqcup)$ is a directoid; if, moreover, $\lambda(x, y) = \lambda(y, x)$ for every pair x, y of A, the directoid is commutative.

Let $(A; \leq, 1)$ be an ordered set with a greatest element 1. For $p \in A$, the interval $[p, 1]$ will be called a *section*. A mapping f of $[p, 1]$ into itself will be called a *sectional mapping*. To distinguish sectional mappings on different sections, we introduce the following notation: if f is a sectional mapping on $[p, 1]$ and $x \in [p, 1]$ then $f(x)$ will be denoted by x^p. A sectional mapping on $[p, 1]$ is called a *switching mapping* if $p^p = 1$ and $1^p = p$ and it is called an *involution* if $x^{pp} = x$ for each $x \in [p, 1]$. Of course, any involution is a bijection and if a sectional mapping on $[p, 1]$ is a switching involution then

$$x^p = 1 \text{ iff } x = p \quad \text{ and } \quad x^p = p \text{ iff } x = 1.$$

$(A; \leq, 1)$ will be called *with sectionally switching involutions* if there is a sectional switching involution on the section $[p, 1]$ for each $p \in A$.

The concept of implication algebra was introduced by J. C. Abbott [1]. It is a groupoid $\mathcal{A} = (A; \circ)$ with a distinguished element 1 (which is an algebraic constant, namely $\mathcal{A}$ satisfies $x \circ x = 1$) in which an order $\leq$ can be induced by $x \leq y$ if and only if $x \circ y = 1$. It was shown [1] that $(A; \leq)$ is a semilattice with a greatest element 1 where $x \vee y = (x \circ y) \circ y$ and, moreover, every section $[p, 1]$ is equipped by a sectional antitone involution $x^p = x \circ p$ (which is in fact a complementation in this section). This concept was generalized in [2] and applied in [3] for axiomatization of logical connective implication in many-valued logics. Let us note the name implication algebra express the fact that $x \circ y$ is interpreted as a connective implication $x \Rightarrow y$.

Lemma 1 *Let $\mathcal{A} = (A; \circ, 1)$ be an algebra of type $(2, 0)$ satisfying the following conditions*

(A1) $x \circ x = 1$, $x \circ 1 = 1$*;*

(A2) $x \circ y = 1$ *implies* $y = (y \circ x) \circ x$*;*

(A3) $x \circ ((((x \circ y) \circ y) \circ z) \circ z) = 1$.

Define a binary relation $\leq$ on A by the setting

$$x \leq y \text{ if and only if } x \circ y = 1. \qquad (*)$$

Then $(A; \leq)$ is an ordered set with a greatest element 1 where for each $p \in A$ the mapping $x \mapsto x^p = x \circ p$ is a sectional switching involution on $[p, 1]$.

Proof By (A1) and (A2) we infer immediately

$$1 \circ x = (x \circ x) \circ x = x. \qquad (**)$$

Due to (A1), the relation $\leq$ is reflexive and $x \leq 1$ for each $x \in A$. Suppose $x \leq y$ and $y \leq x$. Then $x \circ y = 1$, $y \circ x = 1$ and, by (A2), $y = (y \circ x) \circ x = 1 \circ x = x$ thus $\leq$ is antisymmetrical. Suppose $x \leq y$ and $y \leq z$. Then $x \circ y = 1$, $y \circ z = 1$ and by (A1) and (A3) we have

$$x \circ z = x \circ (1 \circ z) = x \circ ((y \circ z) \circ z) = x \circ (((1 \circ y) \circ z) \circ z) = x \circ ((((x \circ y) \circ y) \circ z) \circ z) = 1$$

thus $x \leq z$ proving transitivity of $\leq$.

Now, let $p \in A$ and $x \in [p, 1]$. Then $p \leq x$ and hence $p \circ x = 1$. Due to (A2) we conclude $x^{pp} = (x \circ p) \circ p = x$ thus every sectional mapping $x \mapsto x^p = x \circ p$ is an involution on $[p, 1]$. Applying (A1) and $(**)$ we infer that it is a switching mapping. □

Lemma 2 *Let $\mathcal{A} = (A; \circ, 1)$ satisfy* (A1), (A2), (A3) *and*

(A4) $y \circ (x \circ y) = 1$;

(A5) $x \circ ((x \circ y) \circ y) = 1$.

Then $(x \circ y) \circ y \in U(x, y)$ for each $x, y \in A$.

Proof By Lemma 1, $\leq$ defined by $(*)$ is an order on A. Replace x by $x \circ y$ in (A4) we obtain $y \circ ((x \circ y) \circ y) = 1$ thus $y \leq (x \circ y) \circ y$. By (A5) we have $x \leq (x \circ y) \circ y$ thus $(x \circ y) \circ y \in U(x, y)$. □

Since every implication algebra in the sense of [1] satisfies (A1)–(A5), it motivates us to introduce the following concept: An algebra $\mathcal{A} = (A; \circ, 1)$ satisfying (A1)–(A5) will be called a *weak d−implication algebra.* We can state

Theorem 1 *Let $\mathcal{A} = (A; \circ, 1)$ be a weak d-implication algebra. Define a binary operation $\sqcup$ on A by*

$$x \sqcup y = (x \circ y) \circ y$$

and for each $p \in A$ define $x^p = x \circ p$. Then $\mathcal{D}(A) = (A; \sqcup)$ is a directoid with the greatest element 1 with sectionally switching involutions whose induced order coincides with that of $\mathcal{A}$.

Proof Define $x \sqcup y = (x \circ y) \circ y$ and $x^p = x \circ p$, for $x \in [p, 1]$.
(a) Let $x \circ y = 1$. Then $x \sqcup y = (x \circ y) \circ y = 1 \circ y = y$.
(b) Let $\leq$ be the induced order on $\mathcal{A}$. By (A4) we have $x \circ y \in [y, 1]$. Suppose now $x \sqcup y = y$. Then, since the sectional mapping on $[y, 1]$ is an involution, we infer

$$x \circ y = (x \circ y)^{yy} = ((x \circ y) \circ y) \circ y = (x \sqcup y) \circ y = y \circ y = 1.$$

We have shown $x \circ y = 1$ if and only if $x \sqcup y = y$ thus the order on $\mathcal{A}$ defined by $(*)$ coincides with that of $(A; \sqcup)$ defined by (R). The fact that $(A; \sqcup)$ is a directoid follows directly by Lemma 2 and the fact that $x \leq y$ gets $x \sqcup y = (x \circ y) \circ y = 1 \circ y = y$ and, by (A2), also $y \sqcup x = (y \circ x) \circ x = y$. By Lemma 1, sectional mappings $x \mapsto x^p$ for $x \in [p, 1]$ are switching involutions. □

Example 2 Let $A = \{a, b, c, d, 1\}$ and the operation "$\circ$" on A is given by the table

$\circ$	a	b	c	d	1
a	1	c	1	1	1
b	d	1	1	1	1
c	d	d	1	d	1
d	c	c	c	1	1
1	a	b	c	d	1

One can easily verify the conditions (A1) – (A5) thus $\mathcal{A} = (A; \circ, 1)$ is a weak d-implication algebra. For $\sqcup$ defined by $x \sqcup y = (x \circ y) \circ y$ we obtain just the directoid depicted in Example 1.

To show that directoids with sectional switching involutions can be represented by weak d-implication algebras, we need to prove the converse of Theorem 1.

Theorem 2 *Let $\mathcal{D} = (D; \sqcup, 1)$ be a directoid with a greatest element 1, $\leq$ its induced order. Let for each $p \in D$ there exists a sectional switching involution $x \mapsto x^p$ on $[p, 1]$. Define*

$$x \circ y = (x \sqcup y)^y.$$

Then $\mathcal{A}(D) = (D; \circ, 1)$ is a weak d-implication algebra.

Proof Since $y \leq x \sqcup y$ in $\mathcal{D}$, we have $x \sqcup y \in [y, 1]$ and hence the definition of the new operation "$\circ$" is sound. Moreover, $(x \circ y) \circ y = (x \sqcup y)^{yy} = x \sqcup y$.

We have to verify the conditions (A1)–(A5).

(A1): $x \circ x = (x \sqcup x)^x = x^x = 1$ and $x \circ 1 = (x \sqcup 1)^1 = 1^1 = 1$.

(A2): Suppose $x \circ y = 1$. Then $(x \sqcup y)^y = 1$ thus (since the sectional mapping is a switching bijection) also $x \sqcup y = y$. Conversely, if $x \sqcup y = y$ then $x \circ y = 1$, i.e. the order induced on $\mathcal{D}$ coincides with that given by $(*)$ in Theorem 1. Hence, if $x \circ y = 1$ then $x \leq y$ thus $y \in [x, 1]$, i.e. $(y \circ x) \circ x = y^{xx} = y$.

(A3): By (D4) we have $x \leq (x \sqcup y) \sqcup z$ thus

$$x \circ ((((x \circ y) \circ y) \circ z) \circ z) = x \circ ((x \sqcup y) \sqcup z) = 1.$$

(A4): Since $x \sqcup y \in [y, 1]$, we have $x \circ y = (x \sqcup y)^y \in [y, 1]$ thus $y \leq x \circ y$ whence $y \circ (x \circ y) = 1$.

(A5): Since $y \leq x \sqcup y$ we have

$$(x \circ y) \circ y = ((x \sqcup y)^y \sqcup y)^y = (x \sqcup y)^{yy} = x \sqcup y.$$

Thus $x \leq x \sqcup y = (x \circ y) \circ y$ proving $x \circ ((x \circ y) \circ y) = 1$. □

In what follows, we modify our results for commutative directoids. For this, define a one more concept.

An algebra $\mathcal{A} = (A; \circ, 1)$ of type (2,0) is called a *d-implication algebra* if it satisfies the identities (A1), (A3) and

(B1) $(x \circ y) \circ y = (y \circ x) \circ x$;

(B2) $((x \circ y) \circ y) \circ y = x \circ y$.

The fact that every d-implication algebra is also a weak d-implication algebra will be clear from the next theorems. Let us only mention that d-implication algebras are determined by identities and hence they form a variety.

Lemma 3 *Let $\mathcal{A} = (A; \circ, 1)$ be a d-implication algebra. Define a binary relation $\leq$ on A by the setting $x \leq y$ if and only if $x \circ y = 1$. Then $\leq$ is an order on A and 1 is a greatest element.*

Proof By (A1), $\leq$ is reflexive. Suppose $x \leq y$ and $y \leq x$. Then $x \circ y = 1$, $y \circ x = 1$ and, due to (B1), also $x = 1 \circ x = (y \circ x) \circ x = (x \circ y) \circ y = 1 \circ y = y$, i.e. $\leq$ is antisymmetrical. Transitivity of $\leq$ can be shown identically as in the proof of Lemma 1. By (A1), $x \leq 1$ for each $x \in A$. □

Theorem 3 *Let $\mathcal{A} = (A; \circ, 1)$ be a d-implication algebra. Define*

$$x \sqcup y = (x \circ y) \circ y$$

and for $x \in [y, 1]$ let $x^y = x \circ y$. Then $\mathcal{C}(A) = (A; \sqcup)$ is a commutative directoid with a greatest element 1 and with sectionally switching involutions.

Proof By Lemma 3, $(A; \leq)$ is an ordered set where $x \leq y$ if and only if $x \circ y = 1$ and 1 is a greatest element of $(A; \leq)$. Due to (B1) we infer $x \sqcup y = y \sqcup x$.

By (B1) and (A3) we have

$$x \circ (x \sqcup y) = x \circ ((x \circ y) \circ y) = x \circ ((((x \circ y) \circ y) \circ y) \circ y) = 1$$

thus $x \leq x \sqcup y$. Analogously $y \leq y \sqcup x = x \sqcup y$ thus $x \sqcup y \in U(x, y)$. Further, if $x \leq y$ then

$$x \sqcup y = (x \circ y) \circ y = 1 \circ y = y.$$

We have shown that $(A; \sqcup)$ is a commutative directoid. Analogously as in the previous proofs, the induced order of $(A; \sqcup)$ coincides with $\leq$. Hence, 1 is a greatest element of $(A; \sqcup)$.

Now, let $y \in A$ and $x \in [y, 1]$. Then $y \leq x$ and hence $x^{yy} = (x \circ y) \circ y = x \sqcup y = x$. Further, $y^y = y \circ y = 1$ and $1^y = 1 \circ y = y$ thus for each $y \in A$ the mapping $x \mapsto x^y$ is a sectional switching involution on $[y, 1]$. □

Theorem 4 *Let $\mathcal{C} = (C; \sqcup, 1)$ be a commutative directoid with a greatest element 1. Let $\leq$ be its induced order and for each $p \in C$ there exists a sectional switching involution $x \mapsto x^p$ on $[p, 1]$. Define*

$$x \circ y = (x \sqcup y)^y.$$

Then $\mathcal{A}(C) = (C; \circ, 1)$ is a d-implication algebra.

Proof It was shown in Theorem 2 that " $\circ$ " is correctly defined operation on C satisfying (A1) and (A3), and that $(x \circ y) \circ y = x \sqcup y$. Since $x \sqcup y = y \sqcup x$, (B1) is evident. It remains to prove (B2). Since $y \leq x \sqcup y$, we derive

$$((x \circ y) \circ y) \circ y = (x \sqcup y) \circ y = (x \sqcup y)^y = x \circ y.$$ □

Remark 1 Let $\mathcal{A} = (A; \circ, 1)$ be a d-implication algebra, $\mathcal{C}(A)$ the induced commutative directoid and $\mathcal{A}(\mathcal{C}(A))$ the induced d-implication algebra. Denote by $\bullet$ the binary operation in $\mathcal{A}(\mathcal{C}(A))$. Then

$$x \bullet y = (x \sqcup y)^y = ((x \circ y) \circ y) \circ y = x \circ y$$

by (B2) thus $\mathcal{A}(\mathcal{C}(A)) = \mathcal{A}$.

Remark 2 Let $\mathcal{C} = (C; \sqcup, 1)$ be a commutative directoid with 1 and with sectionally switching involutions. Let $\mathcal{A}(C)$ be the induced d-implication algebra and $\mathcal{C}(\mathcal{A}(C))$ the induced directoid. Denote by $\cup$ the binary operation in $\mathcal{C}(\mathcal{A}(C))$. Since $x \sqcup y \in [y, 1]$, we derive

$$x \cup y = (x \circ y) \circ y = ((x \sqcup y)^y \sqcup y)^y = (x \sqcup y)^{yy} = x \sqcup y$$

thus also $\mathcal{C}(\mathcal{A}(C)) = \mathcal{C}$.

Remark 3 Hence, the mutual correspondence between commutative directoids with 1 and with sectional switching involutions and d-algebras is one-to-one and hence every such $\mathcal{C}$ can be identify with $\mathcal{A}(\mathcal{C})$. However, d-implication algebras form a variety thus also the induced commutative directoids.

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A Decomposition of Homomorphic Images of Nearlattices*

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Abstract

By a nearlattice is meant a join-semilattice where every principal filter is a lattice with respect to the induced order. The aim of our paper is to show for which nearlattice $\mathcal{S}$ and its element c the mapping $\varphi_c(x) = \langle x \vee c, x \wedge_p c\rangle$ is a (surjective, injective) homomorphism of $\mathcal{S}$ into $[c) \times (c]$.

Key words: Nearlattice; semilattice; distributive element; pseudocomplement; dual pseudocomplement.

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It is well-known (see e.g. [4]) that if L is a bounded distributive lattice and $c \in L$ has a complement in L then L is isomorphic to the direct product $[c) \times (c]$. On the other hand, if c is not complemented then the mapping $\varphi_c(x) = \langle x \vee c, x \wedge c\rangle$ is still an injective homomorphism of L into the mentioned direct product and one can discuss whether the homomorphic image $\varphi_c(L)$ is a subdirect product of $[c) \times (c]$.

In what follows we generalize this setting for the so-called nearlattices (see [1–3, 5–8]) and we investigate which of these results remain true. It turns out that our task is reasonable only for a class of so-called nested nearlattices.

Definition 1 By a *nearlattice* we mean a semilattice $\mathcal{S} = (S; \vee)$ where for each $a \in S$ the principal filter $[a) = \{x \in S; a \leq x\}$ is a lattice with respect to the induced order $\leq$ of $\mathcal{S}$.

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Remark 1 Since the operation meet is defined only in a corresponding principal filter, we will indicate this fact by indices, i.e. $\wedge_x$ denotes the meet in $[x)$. On the other hand, if $a, b \in [x)$ and $y \leq x$ then $a, b \in [y)$ and $a \wedge_x b = a \wedge_y b$ since both are considered with respect to the same (induced) order $\leq$.

Definition 2 Let $\mathcal{S} = (S; \vee)$ be a nearlattice and $\emptyset \neq A \subseteq S$. A is called a *sublattice* of $\mathcal{S}$ if it is a lattice with respect to the induced order $\leq$ of $\mathcal{S}$.

A sublattice M of a nearlattice $\mathcal{S}$ is called *maximal* if M is not a proper sublattice of another sublattice of $\mathcal{S}$.

Let $\mathcal{S} = (S; \vee)$ be a nearlattice. Denote by $\mathcal{M}_{\mathcal{S}} = \{M_\gamma, \gamma \in \Gamma\}$ the set of all maximal sublattices M_γ of $\mathcal{S}$.

Further, if there exists an element $c \in \bigcap \mathcal{M}_{\mathcal{S}}$, $\mathcal{S}$ will be called a *nested nearlattice*.

Remark 2 a) Every finite nearlattice $\mathcal{S}$ is nested, because $\mathcal{S}$ is a join semilattice with 1 and $1 \in \bigcap M_{\mathcal{S}}$.

b) An example of an infinite nearlattice which is not nested is shown in Fig. 1.

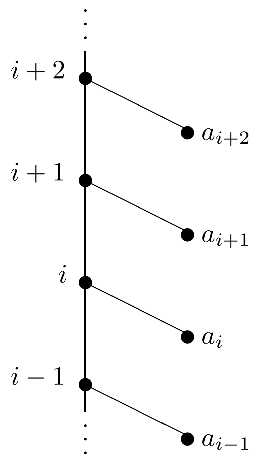

Fig. 1

For any element $c \in S$ we can find a maximal sublattice which does not contain c. In particular, if $c = i$ or $c = a_i$ then c does not belong to the maximal sublattice $[a_{i+1})$.

Let $\mathcal{S}$ be a nested nearlattice and suppose $c \in \bigcap \mathcal{M}_S$. Suppose $x \in \mathcal{S}$. Then there exists $\gamma \in \Gamma$ such that $x \in M_\gamma$. Since M_γ is a lattice and $c \in M_\gamma$, there exists $\inf\{x, c\}$ with respect to the induced order. Suppose $p \in \mathcal{S}$ with $p \leq x, c$. Then clearly $x \wedge_p c = \inf\{x, c\}$. Apparently, this operation does not depend on γ (when x belongs to more than one M_γ). Summarizing, there surely exists $p \in \mathcal{S}$ such that $x \wedge_p c = \inf\{x, c\}$.

Definition 3 Let $\mathcal{S}$ be a nested nearlattice and $c \in \bigcap \mathcal{M}_S$. The mapping $\varphi_c : S \to [c) \times (c]$ defined by

$$\varphi_c(x) = \langle x \vee c, x \wedge_p c \rangle$$

will be called a *decomposition mapping.*

The mapping φ_c is obviously everywhere defined, since $c \in \bigcap \mathcal{M}_{\mathcal{S}}$.

Definition 4 Let $\mathcal{S}$ be a nearlattice and $\{M_\gamma, \gamma \in \Gamma\}$ be the set of its maximal sublattices.

(i) An element a of $\mathcal{S}$ is called *distributive* if

$$a \vee (x \wedge_p y) = (a \vee x) \wedge_p (a \vee y),$$

for all $x, y, p \in M_\gamma$, $p \leq x, y$ and all $\gamma \in \Gamma$.

(ii) An element a is called *dually distributive* if

$$a \wedge_p (x \vee y) = (a \wedge_p x) \vee (a \wedge_p y),$$

for all $a, x, y, p \in M_\gamma$, $p \leq a, x, y$ and all $\gamma \in \Gamma$.

A nearlattice $\mathcal{S}$ is called *distributive* if

$$a \vee (b \wedge_p c) = (a \vee b) \wedge_p (a \vee c)$$

for all $a, b, c \in S$ with $p \leq b, c$.

Suppose now, that an element c is distributive and also dually distributive. We wonder whether φ_c is a homomorphism.

Definition 5 By a *suitable element* we mean an element c of a nested nearlattice $\mathcal{S} = (S; \vee)$ with $c \in \bigcap \mathcal{M}_{\mathcal{S}}$, which is distributive and also dually distributive.

Of course, in a nested distributive nearlattice $\mathcal{S}$ every element $c \in \bigcap \mathcal{M}_{\mathcal{S}}$ is suitable.

Proposition 1 *Let $\mathcal{S} = (S; \vee)$ be a nested nearlattice and c its suitable element. Then the decomposition mapping φ_c is a homomorphism.*

Proof $\varphi_c(x \vee y) = \langle (x \vee y) \vee c, (x \vee y) \wedge_p c \rangle = \langle (x \vee c) \vee (y \vee c), (x \wedge_p c) \vee (y \wedge_p c) \rangle = \langle x \vee c, x \wedge_p c \rangle \vee \langle y \vee c, y \wedge_p c \rangle = \varphi_c(x) \vee \varphi_c(y)$.

$\varphi_c(x \wedge_p y) = \langle (x \wedge_p y) \vee c, (x \wedge_p y) \wedge_p c \rangle = \langle (x \vee c) \wedge_p (y \vee c), (x \wedge_p c) \wedge_p (y \wedge_p c) \rangle = \langle x \vee c, x \wedge_p c \rangle \wedge_p \langle y \vee c, y \wedge_p c \rangle = \varphi_c(x) \wedge_p \varphi_c(y)$. □

Example 1 Let $\mathcal{S}$ be a nearlattice depicted in Fig 2.

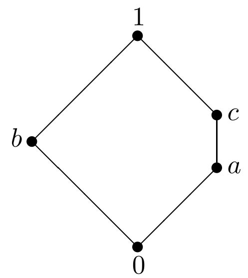

Fig. 2

We can easily check that the elements $0, b, 1$ are distributive and also dually distributive. An element c is distributive, but not dually distributive, an element a is dually distributive, but not distributive.

Consider the decomposition mappings φ_b, φ_a and φ_c. Then for $\varphi_b : \mathcal{S} \mapsto [b) \times (b]$ we have $\varphi_b(1) = \langle 1, b\rangle$, $\varphi_b(0) = \langle b, 0\rangle$, $\varphi_b(b) = \langle b, b\rangle$, $\varphi_b(a) = \langle 1, 0\rangle$ and $\varphi_b(c) = \langle 1, 0\rangle$ (see Fig. 3). Clearly, $[b) = \{b, 1\}$, $(b] = \{0, b\}$.

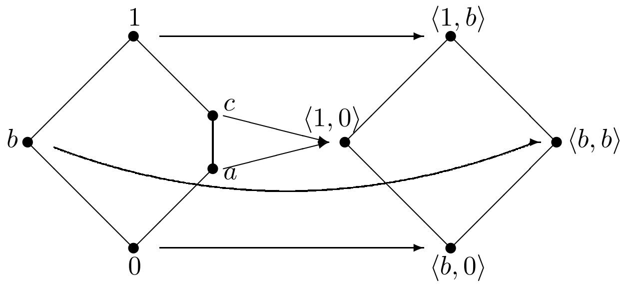

Fig. 3

One can see that the mapping φ_b is a surjective homomorphism which is not injective.

For the decomposition mapping $\varphi_a : \mathcal{S} \mapsto [a) \times (a]$ we have $\varphi_a(1) = \langle 1, a\rangle$, $\varphi_a(0) = \langle a, 0\rangle$, $\varphi_a(a) = \langle a, a\rangle$, $\varphi_a(b) = \langle 1, 0\rangle$ and $\varphi_a(c) = \langle c, a\rangle$ (see Fig. 4). Obviously, $[a) = \{a, c, 1\}$ and $(a] = \{0, a\}$.

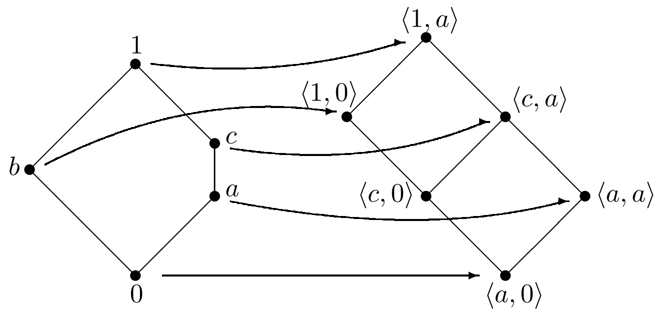

Fig. 4

The mapping φ_a is not a homomorphism, because

$$\varphi_a(c \wedge b) = \langle a, 0\rangle \neq \langle c, 0\rangle = \varphi_a(c) \wedge \varphi_a(b).$$

Similarly, the decomposition mapping $\varphi_c : \mathcal{S} \mapsto [c) \times (c]$ is not a homomorphism.

Now, we will check, whether φ_c is an injection. Let $\varphi_c(x) = \varphi_c(y)$. Then $x \vee c = y \vee c$ and $x \wedge_p c = y \wedge_p c$. If the mapping φ_c is injective, then $x = y$. Thus the mapping φ_c is injective only if for each $x, y \in M_\gamma$ ($x \vee c = y \vee c$ and $x \wedge_p c = y \wedge_p c$) implies $x = y$.

Remark 3 Distributivity and dual distributivity of the element c is not enough to ensure injectivity of the mapping φ_c (see Fig. 3). If we swap b and c, in Fig. 2, we obtain $b \vee c = a \vee c$ and also $b \wedge_0 c = a \wedge_0 c$, but $a \neq b$.

Let us note that for injectivity of φ_c it is not necessary that each maximal sublattice is distributive.

Proposition 2 *If $\mathcal{S} = (S; \vee)$ is a nested distributive nearlattice and $c \in \bigcap \mathcal{M}_\mathcal{S}$, then the decomposition mapping φ_c is injective.*

Proof If $\mathcal{S}$ is distributive then each maximal sublattice is a distributive lattice, in which ($x \vee c = y \vee c$ and $x \wedge_p c = y \wedge_p c$) implies $x = y$. □

If φ_c is an injective homomorphism, then φ_c is an embedding of S into $[c) \times (c]$, i.e. $\mathcal{S}$ is isomorphic to a subnearlattice of this direct product.

Example 2 Denote by $M_1 = \{a, c, 1\}, M_2 = \{b, c, 1\}$ the maximal sublattices of the finite distributive nearlattice $\mathcal{S}$ visualized in Fig. 5.

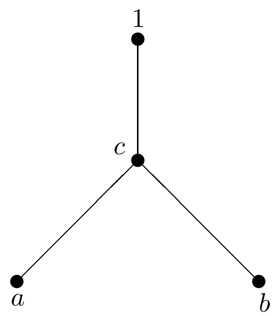

Fig. 5

Evidently $c \in M_1 \cap M_2$. Further, $[c) = \{c, 1\}$ and $(c] = \{c, a, b\}$.

The direct product $[c) \times (c]$ is depicted in Fig. 6.

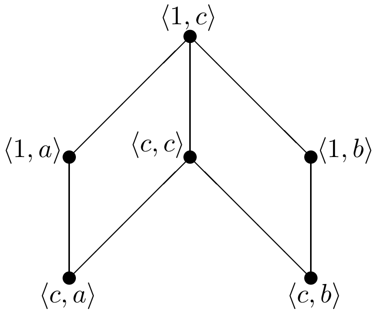

Fig. 6

We have $\varphi_c(1) = \langle 1, c\rangle$, $\varphi_c(c) = \langle c, c\rangle$, $\varphi_c(a) = \langle c, a\rangle$ and $\varphi_c(b) = \langle c, b\rangle$. One can see that φ_c is an injective homomorphism, which is not surjective.

Remark 4 For $c = 1$ (where 1 is the greatest element of S), we obtain: $[c) = \{c\}$, $(c] = S$, and thus $[c) \times (c] \cong \mathcal{S}$.

Now we are interested in assumptions under which the mapping φ_c is surjective. Suppose $\mathcal{S}$ is a nested nearlattice with the set $\{M_\gamma; \gamma \in \Gamma\}$ of its maximal sublattices. An element $a \in S$ has a *complement* b_γ in M_γ if M_γ is a bounded lattice (with 0_γ or 1 as the least or greatest element, respectively) and $a \vee b_\gamma = 1$, $a \wedge_{0_\gamma} b_\gamma = 0_\gamma$.

Proposition 3 *Let $\mathcal{S} = (S; \vee)$ be a nested nearlattice and c its suitable element. Suppose that c has a complement p_γ in each maximal sublattice M_γ of the nearlattice S. Then the decomposition mapping φ_c is a surjective homomorphism.*

Proof We need only to prove, that for each $\langle x, y\rangle \in [c) \times (c]$, there exists an element $z \in S$, such that $\varphi_c(z) = \langle x, y\rangle$.

Since $\langle x, y\rangle \in [c) \times (c]$, then clearly $y \le c \le x$ and there exists $\gamma \in \Gamma$ such that $[y) \subseteq M_\gamma$. Denote by $\wedge_\gamma$ the operation symbol $\wedge_y$ (because it in fact does not depend on y in the following computation).

Take $z = (y \vee p_\gamma) \wedge_\gamma x$. Then

$$\begin{aligned}\varphi_c(z) = \langle z \vee c, z \wedge_\gamma c\rangle &= \langle ((y \vee p_\gamma) \wedge_\gamma x) \vee c, (y \vee p_\gamma) \wedge_\gamma x \wedge_\gamma c\rangle \\ &= \langle (y \vee p_\gamma \vee c) \wedge_\gamma (x \vee c), ((y \wedge_\gamma c) \vee (p_\gamma \wedge_\gamma c)) \wedge_\gamma x\rangle \\ &= \langle (y \vee 1) \wedge_\gamma (x \vee c), ((y \wedge_\gamma c) \vee (p_\gamma \wedge_\gamma c)) \wedge_\gamma x\rangle \\ &= \langle x \vee c, y \wedge_\gamma c \wedge_\gamma x\rangle = \langle x, y\rangle,\end{aligned}$$

proving that φ_c is surjective. □

Corollary 1 *Let $\mathcal{S} = (S; \vee)$ be a nested distributive nearlattice and $\mathcal{M}_\mathcal{S} = \{M_\gamma; \gamma \in \Gamma\}$ the set of its maximal sublattices. If there exists an element $c \in \bigcap \mathcal{M}_\mathcal{S}$ such that c has a complement in each M_γ then the decomposition mapping φ_c is the isomorphism of $\mathcal{S}$ onto $[c) \times (c]$.*

Example 3 The nearlattice S in Fig. 7 is a nested distributive nearlattice which has exactly two distinct maximal sublattices $M_1 = \{a, c, p_1, 1\}$ and $M_2 = \{b, c, p_2, 1\}$. Of course, $c \in M_1 \cap M_2$.

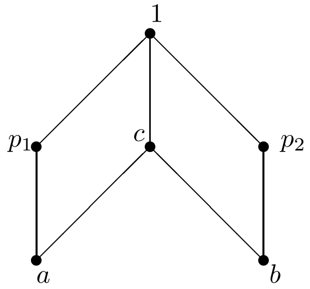

Fig. 7

The complement of c in M_1, is p_1. The complement of c in M_2, is p_2. Clearly, $[c) = \{c, 1\}$, $(c] = \{a, b, c\}$. The direct product $[c) \times (c]$ is depicted in Fig. 6. Obviously, the decomposition mapping $\varphi_c : S \mapsto [c) \times (c]$ is an isomorphism.

Remark 5 If the element c has not a complement in any M_γ, then the mapping φ_c need not be surjective (see Example 2).

Definition 6 Let $(L; \vee, 0, 1)$ be a lattice with the greatest element 1 and the least element 0. An element $c^* \in L$ is called a *pseudocomplement* of $c \in L$, if it is the greatest element such that $c \wedge c^* = 0$. An element $c^+ \in L$ will be called a *dual pseudocomplement* of $c \in L$, if it is the least element for which $c \vee c^+ = 1$.

Proposition 4 *Let $\mathcal{S} = (S; \vee)$ be a nested nearlattice and c its suitable element. Suppose that an element c has a pseudocomplement c^*_γ and a dual pseudocomplement c^+_γ in each maximal sublattice M_γ. Then the homomorphic image $\varphi_c(S)$ is a subdirect product of $[c), (c]$.*

Proof By Proposition 1, φ_c is a homomorphism of $\mathcal{S}$ into $[c) \times (c]$, thus $\varphi_c(S)$ is a subnearlattice of the nearlattice $[c) \times (c]$. We need only to prove that φ_c is surjective in the both components. Let $\langle x, y\rangle \in [c) \times (c]$, i.e. $y \leq c \leq x$. By the assumption, there exist $c^*_\gamma, c^+_\gamma \in M_\gamma$.

Put $z_1 = (y \vee c^+_\gamma) \wedge_\gamma x$. Then

$$\varphi_c(z_1) = \langle z_1 \vee c, z_1 \wedge_\gamma c\rangle = \langle((y \vee c^+_\gamma) \wedge_\gamma x) \vee c, ((y \vee c^+_\gamma) \wedge_\gamma x) \wedge_\gamma c\rangle$$
$$= \langle(y \vee c^+_\gamma \vee c) \wedge_\gamma (x \vee c), ((y \wedge_\gamma c) \vee (c^+_\gamma \wedge_\gamma c)) \wedge_\gamma x\rangle = \langle x, y \vee (c^+_\gamma \wedge_\gamma c)\rangle,$$

thus $\varphi_c(z_1)$ is surjective in the first component.

Consider $z_2 = (y \vee c^*_\gamma) \wedge_\gamma x$. Then

$$\varphi_c(z_2) = \langle z_2 \vee c, z_2 \wedge_\gamma c\rangle = \langle((y \vee c^*_\gamma) \wedge_\gamma x) \vee c, ((y \vee c^*_\gamma) \wedge_\gamma x) \wedge_\gamma c\rangle$$
$$= \langle(y \vee c^*_\gamma \vee c) \wedge_\gamma (x \vee c), ((y \wedge_\gamma c) \vee (c^*_\gamma \wedge_\gamma c)) \wedge_\gamma x\rangle = \langle(c \vee c^*_\gamma) \wedge_\gamma x, y\rangle,$$

i.e. $\varphi_c(z_2)$ is surjective in the second component. □

On the other hand, we are able to get a surjective mapping of $S \times S$ onto $[c) \times (c]$ for a nested nearlattice S and its suitable element c which need not be a homomorphism.

Proposition 5 *Let $\mathcal{S} = (S; \vee)$ be a nested nearlattice and c its suitable element. Suppose that an element c has a pseudocomplement c^*_γ and a dual pseudocomplement c^+_γ in each maximal sublattice M_γ. Denote by ψ_c a mapping from $S \times S$ into $[c) \times (c]$, defined by*

$$\psi_c(z_1, z_2) = \langle z_1 \vee c, z_2 \wedge_\gamma c \rangle,$$

where $\gamma \in \Gamma$, such that $z_2 \in M_\gamma$. Then ψ_c is a surjective mapping of $S \times S$ onto $[c) \times (c]$.

Proof Let $\langle x, y \rangle \in [c) \times (c]$, then $y \leq c \leq x$. Hence there exists $\gamma \in \Gamma$ such that $[y) \subseteq M_\gamma$.

Take $z_1 = (y \vee c^+_\gamma) \wedge_\gamma x$, $z_2 = (y \vee c^*_\gamma) \wedge_\gamma x$. Then

$$\begin{aligned} \psi_c(z_1, z_2) &= \langle z_1 \vee c, z_2 \wedge_\gamma c \rangle \\ &= \langle ((y \vee c^+_\gamma) \wedge_\gamma x) \vee c, ((y \vee c^*_\gamma) \wedge_\gamma x) \wedge_\gamma c \rangle \\ &= \langle (y \vee c^+_\gamma \vee c) \wedge_\gamma (x \vee c), ((y \wedge_\gamma c) \vee (c^*_\gamma \wedge_\gamma c)) \wedge_\gamma x \rangle \\ &= \langle x \vee c, y \wedge_\gamma c \rangle = \langle x, y \rangle, \end{aligned}$$

thus ψ_c is a surjective mapping of $S \times S$ onto $[c) \times (c]$. □

We finish with a note concerning lattices.

Remark 6 Let $\mathcal{L} = (L; \vee, \wedge)$ be a bounded lattice and suppose that an element $c \in \mathcal{L}$ has a pseudocomplement c^* and a dual pseudocomplement c^+. Let the elements c^+ and c^* are distributive and dually distributive. Introduce a mapping:

$$\psi_{c^+,c^*} : L \mapsto [c^+) \times (c^*], \psi_{c^+,c^*}(z) = \langle z \vee c^+, z \wedge c^* \rangle.$$

Since the decomposition mappings φ^*_c and φ^+_c are homomorphisms by Proposition 1, also ψ_{c^+,c^*} is a homomorphism.

Further, analogously as in the Proposition 4 and the Proposition 5, it is easy to show that the mapping φ_{c^+} is surjective in the first component, the mapping φ_{c^*} is surjective in the second component and the mapping ψ_{c^+,c^*} is a surjective homomorphism of the lattice $\mathcal{L}$ onto $[c^+) \times (c^*]$.

Example 4 Let $\mathcal{L}$ be the eight element lattice depicted on the left hand side in Fig. 8.

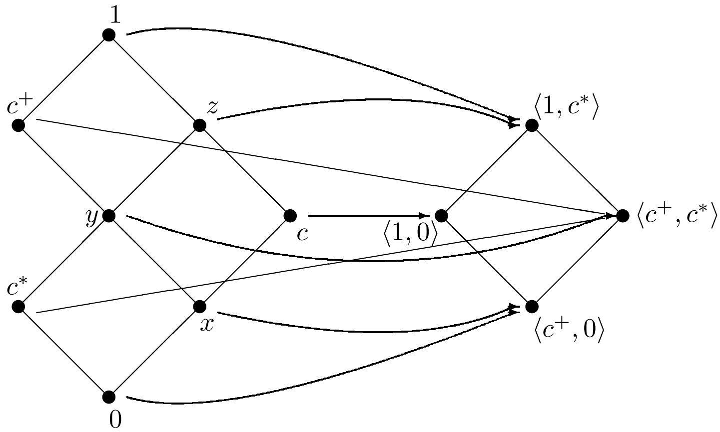

Fig. 8

$\mathcal{L}$ is obviously distributive. Clearly $[c^+) = \{c^+, 1\}$ and $(c^*] = \{0, c^*\}$ (see the lattice $(c^+] \times (c^*]$ on the right hand side of Fig. 8). The mapping ψ_{c^+,c^*} is a surjective homomorphism of the lattice $\mathcal{L}$ onto $[c^+) \times (c^*]$, given by

$$\psi_{c^+,c^*}(1) = \psi_{c^+,c^*}(z) = \langle 1, c^* \rangle,$$

$$\psi_{c^+,c^*}(c^+) = \psi_{c^+,c^*}(y) = \psi_{c^+,c^*}(c^*) = \langle c^+, c^* \rangle,$$

$$\psi_{c^+,c^*}(c) = \langle 1, 0 \rangle,$$

$$\psi_{c^+,c^*}(x) = \psi_{c^+,c^*}(0) = \langle c^+, 0 \rangle.$$

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Direct Decompositions and Basic Subgroups in Commutative Group Rings

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Abstract

An attractive interplay between the direct decompositions and the explicit form of basic subgroups in group rings of abelian groups over a commutative unitary ring are established. In particular, as a consequence, we give a simpler confirmation of a more general version of our recent result in this aspect published in Czechoslovak Math. J. (2006).

Key words: Direct decompositions; basic subgroups; normed units; group rings.

2000 Mathematics Subject Classification: 16U60, 16S34; 20K10, 20K21

1 Introduction

Throughout the text of this brief paper, let G be an abelian group with p-component G_p, written by multiplicative record, and R a commutative ring with identity (of prime characteristic, for instance p, for applications). As usual, RG denotes the group ring of G over R with group of normalized units $V(RG)$, abbreviated for facilitating of the exposition via $V(G)$. For a subgroup A of G, we define by the same reason $I(G; A)$ as the relative augmentation ideal of RG with respect to A. All other notation and terminology from the abelian group theory, not expressly given here, are either standard or follow the classical book of Fuchs [5].

Nachev has demonstrated in [6] that there is a transversal between the basic subgroups of G and $V(G)$, provided that G is p-primary and R is p-perfect with prime characteristic p. Specifically, Nachev established in an explicit form a series of basic subgroups of $V(G)$ by the usage of the basic subgroups of a p-group G under the limitation on R to be p-perfect of prime characteristic p. In the proof, he uses intensively a relationship between the properties of a fixed basic subgroup B of G and the decomposition of $(1 + I(G; B)) \cap V(G)_p$ into appropriate direct factors.

This approach of finding such a connection also captures the spirit of the present short note. We thus identify and focus on certain suitable decompositions in $V(G)_p$ by developing the Nachev's method to the mixed case. Thereby we state and prove many of the results in the more general setting of arbitrary groups. We hereafter will also accent on the kind of a basic subgroup of $V(G)_p$ over a coefficient ring larger than the corresponding one in [4].

2 Main Results

We first hasten to the following key technicality (see [1] and [3] too). It is worthwhile noticing that it encompasses Lemma 7 from [6].

Lemma 1 ([2], Lemma 6) *Assume that $G = C \times A$ is an abelian group and R is any commutative unitary ring. Then $V(G)_p = V(C)_p \times [(1+I(G;A)) \cap V(G)_p]$.*

We are now ready to proceed by proving the following formula which is a non-trivial strengthening of formula (11) of the scheme for proof in [6].

Theorem 1 (Decomposition) *Suppose G is an abelian group with a p-basic subgroup B and suppose R is a commutative ring with identity element. Then the following decomposition holds:*

$$(1 + I(G; B)) \cap V(G)_p = \left(1 + I(G; B_0) + \coprod_{n=1}^{\infty} (1 + I(G_{n-1}; B_n))\right) \cap V(G)_p,$$

where $B = \coprod_{n=0}^{\infty} B_n$; $B_n \cong \coprod_{\alpha_n} \langle p^n \rangle$, $\forall n \geq 1$; $B_0 \cong \coprod_{\alpha_0} \mathrm{Z}$ (where α_n is a cardinal $\forall\ n \geq 0$ and Z is an infinite cyclic group) and $G = \coprod_{1 \leq i \leq n} B_i \times G_n$ with $G_n = B_{n+1} \times G_{n+1}$ and $G_n = (B_0 \times \coprod_{i=n+1}^{\infty} B_i)G^{p^n}$, $\forall n \geq 1$; $G_0 = G$.

Proof In accordance with [5, p. 163, Theorem 32.4] we subsequently write down $B = \coprod_{n=0}^{\infty} B_n$, $G = \coprod_{1 \leq i \leq n} B_i \times G_n$, $G_n = G_{n+1} \times B_{n+1}$ and $G_n = B_n^* G^{p^n}$, where $B_n^* = B_0 \times \coprod_{i=n+1}^{\infty} B_i$. It is worth to noting that B_0 is not in general a direct factor of G.

Employing the foregoing Lemma 1 for $n = 1$ we derive that

$$V(G)_p = V(G_1)_p \times [(1 + I(G; B_1)) \cap V(G)_p].$$

Consequently, by induction on n we obtain

$$V(G)_p = V(G_n)_p \times \Big[\coprod_{1\le i\le n} (1 + I(G_{i-1}; B_i)) \cap V(G)_p\Big],$$

where $G_0 = G$. Furthermore, we observe that

$$\Big[\coprod_{1\le i\le n} (1 + I(G_{i-1}; B_i))\Big] \cap V(G)_p \subseteq \Big(1 + I\Big(G; \coprod_{1\le i\le n} B_i\Big)\Big) \cap V(G)_p \le V(G)_p.$$

Therefore, the previous decomposition implies that

$$\Big(1 + I\Big(G; \coprod_{1\le i\le n} B_i\Big)\Big) \cap V(G)_p =$$

$$= \Big[\coprod_{1\le i\le n} (1 + I(G_{i-1}; B_i)) \cap V(G)_p\Big] \times \Big(V(G_n)_p \cap \Big(1 + I\Big(G; \coprod_{1\le i\le n} B_i\Big)\Big)\Big).$$

But $G_n \cap \coprod_{1\le i\le n} B_i = 1$, so the latter intersection is equal to 1 (e.g. [1]). Thus the last decomposition transforms to the following:

$$\Big(1 + I\Big(G; \coprod_{1\le i\le n} B_i\Big)\Big) \cap V(G)_p = \Big(\coprod_{1\le i\le n} (1 + I(G_{i-1}; B_i))\Big) \cap V(G)_p, \quad \forall n \ge 1.$$

Finally, since B is the union of an ascending chain of subgroups $B_0 \times \coprod_{1\le i\le n} B_i$ ($n = 1, 2, \ldots$), whence because of the finite support $(1 + I(G; B)) \cap V(G)_p$ is the union of

$$\Big(1+I\Big(G; B_0\times \coprod_{1\le i\le n} B_i\Big)\Big)\cap V(G)_p = \Big(1+I(G; B_0)+I\Big(G; \coprod_{1\le i\le n} B_i\Big)\Big)\cap V(G)_p,$$

by taking in both sides of the last identity the limit operation $n \to \infty$, we deduce that

$$\Big(1 + I\Big(G; \coprod_{i=0}^{\infty} B_i\Big)\Big) \cap V(G)_p = \Big(1 + I(G; B_0) + \coprod_{i=1}^{\infty}(1 + I(G_{i-1}; B_i))\Big) \cap V(G)_p,$$

which is precisely the desired equality. □

We are now in a position to give an alternative verification of the following assertion (see, for instance, [4, Theorem 2]).

Theorem 2 (Basis) *Let G be an abelian group with a p-basic subgroup B and let R be a perfect commutative ring with 1 of prime characteristic p. Then $[1 + I(G; B)] \cap V(G)_p$ is a basic subgroup of $V(G)_p$.*

Proof In order to show the truthfulness of this claim, it is enough to check only the validity of three conditions from the definition of a basic subgroup (see, for example, [5]).

1) The fact that $[1+I(G;B)]\cap V(G)_p$ is a coproduct of cyclic groups follows at once by the method described in [4] and applied to R and B_0 as well as by the preceding Theorem, because $B_n^{p^n}=1$ forces that

$$(1+I(G_{n-1};B_n))^{p^n}=1+I^{p^n}(G_{n-1};B_n)=1+I(G_{n-1}^{p^n};B_n^{p^n})=1.$$

2) The property of $[1+I(G;B)]\cap V(G)_p$ to be a pure subgroup of $V(G)_p$ follows like this: For each $n\geq 1$ we calculate with the aid of [1] that

$$\begin{aligned}[(1+I(G;B))\cap V(G)_p]\cap V(G)_p^{p^n} &= (1+I(G;B))\cap V(G^{p^n})_p\\ = (1+I(G^{p^n};B^{p^n}))\cap V(G^{p^n})_p &= (1+I(G;B)^{p^n})\cap V(G)_p^{p^n}\\ = (1+I(G;B))^{p^n}\cap V(G)_p^{p^n} &= [(1+I(G;B))\cap V(G)_p]^{p^n}.\end{aligned}$$

3) The divisibility of the quotient group $V(G)_p/[(1+I(G;B))\cap V(G)_p]$ can be verified as follows: Writing $G=BG^p$, and taking into account that $G_p=B_pG_p^p$, we conclude by application of the main proposition in [3] that

$$V(G)_p=V(G^p)_p[(1+I(G;B))\cap V(G)_p].$$

Since $V(G^p)_p=V(G)_p^p$, we are done. □

Remark 1 In [4] the same affirmation as alluded to above was proved under the more restrictive assumption on R to be a field. The foregoing theorem extends this result to an arbitrary commutative unitary ring R. Besides, the idea used here is at all different to that in [4].

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Two Different However Equivalent Methods for Derivation of Estimators of Parameters in Deformation Measurements

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Abstract

The aim of this paper is to develop two different methods for an executing of the deformation measurement and to prove that these two methods are equivalent which is a advantage for a conclusive verification of the results of the experiment in a practice.

Key words: Multiepoch linear model, multivariate regression model with constraints.

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1 Introduction

The aim is to develop two different methods for an executing of four epochs experiment in which the movements of the reference points on a dam during the gradual filling of the dam have been measured. According to the instructions of a structural designer these points should move along the specific trajectories. The aim of this experiment is to compare these theoretical trajectories with empirical ones. In the first method coordinates of the reference points and the parameters that describe trajectories of these points are estimated at the same time. In the second method the coordinates of the points are estimated first of all and these estimates are used for a calculation of the trajectories. The corrected coordinates from the second method must be equal to the estimated coordinates from the first method. The first procedure can be realized after realization of the 4th epoch measurement only. Since it is necessary to know preliminary results (shifts of the reference points) during the single epoch, we must estimate coordinates after each epoch separately. At the end of the 4th epoch estimations of the coordinates of the all reference points are at our disposal and the parameters of the trajectories can be estimated by means of second method. However at the same time both coordinates and trajectories parameters can be estimated simultaneously in another, however equivalent model (first method). Both methods should give the same result for the parameters of the trajectories.

2 Notation and auxiliary statements

Let $\underline{\mathbf{Y}}$ be $n \times m$ random matrix (observation matrix), $\underline{\mathbf{Y}} = (\mathbf{Y}_1, \dots, \mathbf{Y}_m)$, with the mean value $\mathrm{E}(\underline{\mathbf{Y}}) = \mathbf{XB}$. $\mathbf{X}$ is an $n \times k$ given design matrix, $\mathbf{B}$ is an $k \times m$ matrix of unknown parameters (coordinates of the reference points) and $\mathbf{C}$ is an $k \times q$ matrix of unknown parameters (parameters of the trajectories). $\mathbf{I} \otimes \mathbf{\Sigma}$ is the covariance matrix of the observation vector $\mathrm{vec}(\underline{\mathbf{Y}}) = (\mathbf{Y}_1', \dots, \mathbf{Y}_{\mathrm{m}}')'$ and the constraints are given in a form $\mathbf{BH} + \mathbf{CZ} + \mathbf{G} = \mathbf{0}$. Here the matrix $\mathbf{H}, \mathbf{Z}, \mathbf{G}$ are known.

The model

$$\underline{\mathbf{Y}} \sim (\mathbf{XB}, \mathbf{I} \otimes \mathbf{\Sigma}),$$

is regular if $r(\mathbf{X}_{n,k}) = k < n$ and $\mathbf{\Sigma}$ is positive definite. The constraints $\mathbf{BH} + \mathbf{CZ} + \mathbf{G} = \mathbf{0}$ are regular if $r(\mathbf{H}_{m,r}', \mathbf{Z}_{q,r}') = r < m + q$ and $\mathrm{r}(\mathbf{Z}_{q,r}) = q < r$. In the following text it is also assumed $r(\mathbf{H}_{m,r}) = r < m$.

In the following $\mathbf{A}^+$ denotes the Moore–Penrose generalized inverse of the matrix $\mathbf{A}$ (i.e. $\mathbf{AA}^+\mathbf{A} = \mathbf{A}$, $\mathbf{A}^+\mathbf{AA}^+ = \mathbf{A}^+$, $\mathbf{AA}^+ = (\mathbf{AA}^+)'$, $\mathbf{A}^+\mathbf{A} = (\mathbf{A}^+\mathbf{A})'$ cf. [3]).

The symbol $\mathbf{M}_X$ means the projection matrix $\mathbf{I} - \mathbf{P}_X$, where $\mathbf{I}$ is the identity matrix and $\mathbf{P}_X$ is the projection matrix (in the Euclidean norm) on the subspace $\mathcal{M}(\mathbf{X}) = \{\mathbf{Xu} : \mathbf{u} \in \mathrm{R}^k\}$. Here R^k means the k dimensional real vector space.

Lemma 1 *Let the model and the constraints*

$$\underline{\mathbf{Y}} \sim_{nm} (\mathbf{XB}, \mathbf{I} \otimes \mathbf{\Sigma}), \quad \mathbf{B}_{k,m}\mathbf{H}_{m,r} + \mathbf{C}_{k,q}\mathbf{Z}_{q,r} + \mathbf{G} = \mathbf{0}_{k,r} \tag{1}$$

be regular. Then the best linear unbiased estimators (BLUE) of the matrices $\mathbf{B}$ *a* $\mathbf{C}$ *are*

$$\begin{aligned}\widehat{\widehat{\mathbf{B}}} = &-\mathbf{G}\Big[\mathbf{I} - (\mathbf{H}'\mathbf{H})^{-1}\mathbf{Z}'[\mathbf{Z}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{Z}']^{-1}\mathbf{Z}\Big](\mathbf{H}'\mathbf{H})^{-1}\mathbf{H}' + \\ &+ \widehat{\mathbf{B}}\Big[\mathbf{M}_H + \mathbf{H}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{Z}'[\mathbf{Z}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{Z}']^{-1}\mathbf{Z}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{H}'\Big],\end{aligned} \tag{2}$$

$$\widehat{\widehat{\mathbf{C}}} = -\mathbf{G}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{Z}'[\mathbf{Z}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{Z}']^{-1} - \widehat{\mathbf{B}}\mathbf{H}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{Z}'[\mathbf{Z}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{Z}']^{-1} \tag{3}$$

and

$$\mathrm{Var}[\mathrm{vec}(\hat{\hat{\mathbf{B}}})] =$$
$$= \Big[\mathbf{M}_H + \mathbf{H}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{Z}'[\mathbf{Z}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{Z}']^{-1}\mathbf{Z}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{H}'\Big] \otimes (\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1},$$
$$\mathrm{Var}[\mathrm{vec}(\hat{\hat{\mathbf{C}}})] = [\mathbf{Z}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{Z}']^{-1} \otimes (\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1}.$$

Here $\widehat{\mathbf{B}} = (\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{\Sigma}^{-1}\underline{\mathbf{Y}}$.

Proof In the univariate regular model

$$\mathbf{Y} \sim (\mathbf{X}\boldsymbol{\beta}_1, \mathbf{\Sigma}), \quad \mathbf{B}_1\boldsymbol{\beta}_1 + \mathbf{B}_2\boldsymbol{\beta}_2 + \mathbf{b} = \mathbf{0},$$

the BLUE of $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ are

$$\begin{pmatrix} \hat{\hat{\boldsymbol{\beta}}}_1 \\ \hat{\hat{\boldsymbol{\beta}}}_2 \end{pmatrix} = -\begin{pmatrix} (\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{B}_1'\mathbf{Q}_{1,1} \\ \mathbf{Q}_{2,1} \end{pmatrix}\mathbf{b} + \begin{pmatrix} \mathbf{I} - (\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{B}_1'\mathbf{Q}_{1,1}\mathbf{B}_1 \\ -\mathbf{Q}_{2,1}\mathbf{B}_1 \end{pmatrix}\hat{\boldsymbol{\beta}}_1$$

and

$$\begin{aligned} \mathrm{var}(\hat{\hat{\boldsymbol{\beta}}}_1) &= (\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1} - (\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{B}_1'\mathbf{Q}_{1,1}\mathbf{B}_1(\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1}, \\ \mathrm{var}(\hat{\hat{\boldsymbol{\beta}}}_2) &= -\mathbf{Q}_{2,2}, \end{aligned}$$

where $\hat{\boldsymbol{\beta}}_1 = (\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{Y}$ and

$$\begin{pmatrix} \mathbf{Q}_{1,1}, & \mathbf{Q}_{1,2} \\ \mathbf{Q}_{2,1}, & \mathbf{Q}_{2,2} \end{pmatrix} = \begin{pmatrix} \mathbf{B}_1(\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{B}_1', & \mathbf{B}_2 \\ \mathbf{B}_2', & \mathbf{0} \end{pmatrix}^{-1}$$
$$= \begin{pmatrix} \{\mathbf{M}_{B_2}\mathbf{A}\mathbf{M}_{B_2}\}^+, & (\mathbf{B}_2')^-_{m(A)} \\ \left[(\mathbf{B}_2')^-_{m(A)}\right]', & -\left[(\mathbf{B}_2')^-_{m(A)}\right]'\mathbf{A}(\mathbf{B}_2')^-_{m(A)} \end{pmatrix}.$$

Here $\mathbf{A} = \mathbf{B}_1(\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{B}_1'$ and $(\mathbf{B}_2')^-_{m(A)}$ denotes minimum $\mathbf{A}$-seminorm generalized inverse of the matrix $\mathbf{B}_2'$. (cf. theory of the Pandora-Box matrix in [3])

Now it suffices to write the multivariate model in the form

$$\mathrm{vec}(\underline{\mathbf{Y}}) \sim [(\mathbf{I} \otimes \mathbf{X})\mathrm{vec}(\mathbf{B}), \mathbf{I} \otimes \mathbf{\Sigma}],$$
$$(\mathbf{H}' \otimes \mathbf{I})\mathrm{vec}(\mathbf{B}) + (\mathbf{Z}' \otimes \mathbf{I})\mathrm{vec}(\mathbf{C}) + \mathrm{vec}(\mathbf{G}) = \mathbf{0}$$

and use the equalities

$$\begin{aligned}
\mathbf{Q}_{1,1} &= \left\{\mathbf{M}_{(Z'\otimes I)}[(\mathbf{H}'\mathbf{H})\otimes(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}]\mathbf{M}_{(Z'\otimes I)}\right\}^{+}\\
&= [(\mathbf{H}'\mathbf{H})^{-1}\otimes(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})] - [(\mathbf{H}'\mathbf{H})^{-1}\otimes(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})](\mathbf{Z}'\otimes\mathbf{I})\\
&\quad\times\left[(\mathbf{Z}\otimes\mathbf{I})[(\mathbf{H}'\mathbf{H})^{-1}\otimes(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})](\mathbf{Z}'\otimes\mathbf{I})\right]^{-1}\\
&\quad\times(\mathbf{Z}\otimes\mathbf{I})[(\mathbf{H}'\mathbf{H})^{-1}\otimes(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})]\\
&= [(\mathbf{H}'\mathbf{H})^{-1}\otimes(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})]\left\{(\mathbf{I}\otimes\mathbf{I}) - \left[\mathbf{Z}'[\mathbf{Z}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{Z}']^{-1}\mathbf{Z}(\mathbf{H}'\mathbf{H})^{-1}\right]\otimes\mathbf{I}\right\},\\
\mathbf{Q}_{2,1} &= \left[(\mathbf{Z}\otimes\mathbf{I})^{-}_{m[(H'H)\otimes(X'\Sigma^{-1}X)^{-1}]}\right]'\\
&= \left[(\mathbf{Z}\otimes\mathbf{I})[(\mathbf{H}'\mathbf{H})^{-1}\otimes(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})](\mathbf{Z}'\otimes\mathbf{I})\right]^{-1}\\
&\quad\times(\mathbf{Z}\otimes\mathbf{I})[(\mathbf{H}'\mathbf{H})^{-1}\otimes(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})] = \left\{[\mathbf{Z}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{Z}']^{-1}\mathbf{Z}(\mathbf{H}'\mathbf{H})^{-1}\right\}\otimes\mathbf{I},\\
\mathbf{Q}_{2,2} &= -\left[(\mathbf{Z}\otimes\mathbf{I})^{-}_{m[(H'H)\otimes(X'\Sigma^{-1}X)^{-1}]}\right]'[(\mathbf{H}'\mathbf{H})\otimes(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}]\\
&\quad\times(\mathbf{Z}\otimes\mathbf{I})^{-}_{m[(H'H)\otimes(X'\Sigma^{-1}X)^{-1}]}\\
&= -\left[(\mathbf{Z}\otimes\mathbf{I})[(\mathbf{H}'\mathbf{H})^{-1}\otimes(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})](\mathbf{Z}'\otimes\mathbf{I})\right]^{-1}\\
&= -[\mathbf{Z}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{Z}']^{-1}\otimes(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}
\end{aligned}$$

and $\mathrm{vec}(\mathbf{AXB}) = (\mathbf{B}'\otimes\mathbf{A})\mathrm{vec}(\mathbf{X})$. □

Lemma 2 *The BLUEs of the matrices* **B** *and* **C** *are the same in the model (1) and in the model*

$$\underline{\widehat{\mathbf{B}}} \sim_{km} [\mathbf{B}, \mathbf{I}\otimes(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}], \quad \mathbf{B}_{k,m}\mathbf{H}_{m,r} + \mathbf{C}_{k,q}\mathbf{Z}_{q,r} + \mathbf{G} = \mathbf{0}_{k,r} \qquad (4)$$

respectively.

Proof We write the model (4) in the form

$$\mathrm{vec}(\underline{\widehat{\mathbf{B}}}) \sim [(\mathbf{I}\otimes\mathbf{I})\mathrm{vec}(\mathbf{B}), \mathbf{I}\otimes(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}],$$
$$(\mathbf{H}'\otimes\mathbf{I})\mathrm{vec}(\mathbf{B}) + (\mathbf{Z}'\otimes\mathbf{I})\mathrm{vec}(\mathbf{C}) + \mathrm{vec}(\mathbf{G}) = \mathbf{0}$$

and use the relations from the proof of Lemma 1

$$\begin{pmatrix}\mathbf{Q}_{1,1}, & \mathbf{Q}_{1,2}\\ \mathbf{Q}_{2,1}, & \mathbf{Q}_{2,2}\end{pmatrix} = \begin{pmatrix}(\mathbf{H}'\otimes\mathbf{I})\{(\mathbf{I}\otimes\mathbf{I})[\mathbf{I}\otimes(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})](\mathbf{I}\otimes\mathbf{I})\}^{-1}(\mathbf{H}\otimes\mathbf{I}), & \mathbf{Z}'\otimes\mathbf{I}\\ \mathbf{Z}\otimes\mathbf{I}, & \mathbf{0}\end{pmatrix}^{-1},$$

$$\begin{aligned}
\mathbf{Q}_{1,1} &= \left\{\mathbf{M}_{(Z'\otimes I)}[(\mathbf{H}'\mathbf{H})\otimes(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}]\mathbf{M}_{(Z'\otimes I)}\right\}^{+}\\
&= [(\mathbf{H}'\mathbf{H})^{-1}\otimes(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})]\left\{(\mathbf{I}\otimes\mathbf{I}) - \left[\mathbf{Z}'[\mathbf{Z}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{Z}']^{-1}\mathbf{Z}(\mathbf{H}'\mathbf{H})^{-1}\right]\otimes\mathbf{I}\right\},\\
\mathbf{Q}_{2,1} &= \left[(\mathbf{Z}\otimes\mathbf{I})^{-}_{m[(H'H)\otimes(X'\Sigma^{-1}X)^{-1}]}\right]' = \left\{[\mathbf{Z}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{Z}']^{-1}\mathbf{Z}(\mathbf{H}'\mathbf{H})^{-1}\right\}\otimes\mathbf{I}
\end{aligned}$$

and

$$\mathbf{Q}_{2,2} = -\left[(\mathbf{Z}\otimes\mathbf{I})^{-}_{m[(H'H)\otimes(X'\Sigma^{-1}X)^{-1}]}\right]'[(\mathbf{H}'\mathbf{H})\otimes(\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1}]$$
$$\times(\mathbf{Z}\otimes\mathbf{I})^{-}_{m[(H'H)\otimes(X'\Sigma^{-1}X)^{-1}]} = -[\mathbf{Z}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{Z}']^{-1}\otimes(\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1}.$$

Thus the corrected estimate $\widehat{\widehat{\mathbf{B}}}$ of the preliminary estimate $\widehat{\mathbf{B}}$ is given by the relation

$$\begin{pmatrix}\mathrm{vec}(\widehat{\widehat{\mathbf{B}}})\\ \mathrm{vec}(\widehat{\widehat{\mathbf{C}}})\end{pmatrix} = -\begin{pmatrix}\{(\mathbf{I}\otimes\mathbf{I})[\mathbf{I}\otimes(\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})](\mathbf{I}\otimes\mathbf{I})\}^{-1}(\mathbf{H}\otimes\mathbf{I})\mathbf{Q}_{1,1}\\ \mathbf{Q}_{2,1}\end{pmatrix}\mathrm{vec}(\mathbf{G})$$
$$+\begin{pmatrix}(\mathbf{I}\otimes\mathbf{I})-\{(\mathbf{I}\otimes\mathbf{I})[\mathbf{I}\otimes(\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})](\mathbf{I}\otimes\mathbf{I})\}^{-1}(\mathbf{H}\otimes\mathbf{I})\mathbf{Q}_{1,1}(\mathbf{H}'\otimes\mathbf{I})\\ -\mathbf{Q}_{2,1}(\mathbf{H}'\otimes\mathbf{I})\end{pmatrix}$$
$$\times\{\mathbf{I}\otimes[(\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{\Sigma}^{-1}]\}\mathrm{vec}(\underline{\mathbf{Y}}),$$

i.e.

$$\widehat{\widehat{\mathbf{B}}} = -\mathbf{G}\left[\mathbf{I}-(\mathbf{H}'\mathbf{H})^{-1}\mathbf{Z}'[\mathbf{Z}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{Z}']^{-1}\mathbf{Z}\right](\mathbf{H}'\mathbf{H})^{-1}\mathbf{H}'$$
$$+\widehat{\mathbf{B}}\left[\mathbf{M}_H+\mathbf{H}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{Z}'[\mathbf{Z}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{Z}']^{-1}\mathbf{Z}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{H}'\right],$$
$$\widehat{\widehat{\mathbf{C}}} = -\mathbf{G}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{Z}'[\mathbf{Z}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{Z}']^{-1}-\widehat{\mathbf{B}}\mathbf{H}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{Z}'[\mathbf{Z}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{Z}']^{-1}.$$

(cf. the relations (2), (3)). □

Remark 1 The analogous lemma for univariate model without constraints cf. [2], p. 398, Theorem 9.2.12.

3 Statistical model of experiment

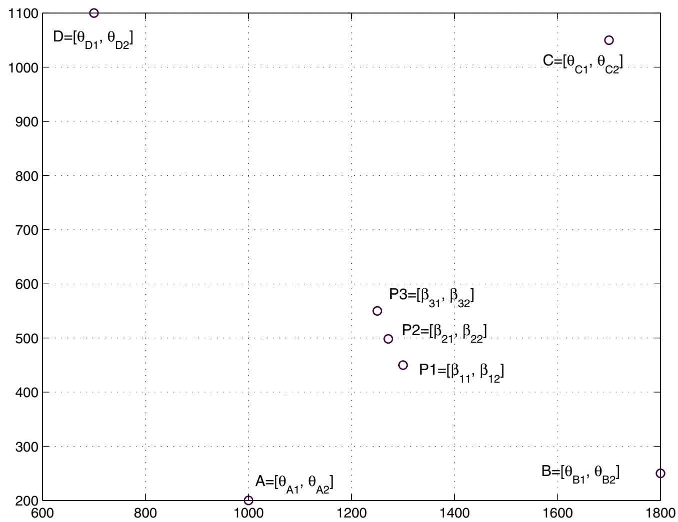

Fig. 1: Position of the points A, B, C, D and the reference points P_1, P_2, P_3.

A deformation measurement is realized according the scheme given by Fig. 1. Here A, B, C, D are points with given coordinates and the reference points are described as P_1, P_2, P_3. The distances are measured in meters with the standard deviation $\sigma_s = 0.01$m and the angles are measured with standard deviation $\sigma_\omega = \frac{3}{206265}$rad. A model of four epochs experiment is considered in the form (1) and (4), where the ith column of $\underline{\mathbf{Y}}$ is the observation vector of the ith epoch, $i = 1, \dots 4$ minus values calculated from approximate coordinates,

$$\mathbf{Y}_i = \begin{pmatrix} \sqrt{(\beta_{11}^{(i)} - \theta_{A1})^2 + (\beta_{12}^{(i)} - \theta_{A2})^2} \\ \vdots \\ \arctan \frac{\beta_{12}^{(i)} - \theta_{A2}}{\beta_{11}^{(i)} - \theta_{A1}} - \arctan \frac{\theta_{B2} - \theta_{A2}}{\theta_{B1} - \theta_{A1}} \\ \vdots \end{pmatrix}, \quad \mathbf{f}_0 = \mathbf{f}(\boldsymbol{\beta}^0).$$

A choice of the approximate coordinates $\boldsymbol{\beta}^0$ is the same for each epoch. Thus the design matrix

$$\mathbf{X} = \frac{\partial \mathbf{E}(\mathbf{Y}_i)}{\partial (\boldsymbol{\beta}^{(i)})'} \bigg|_{\boldsymbol{\beta}^{(i)} = \boldsymbol{\beta}^0}$$

is common for all epochs.

Estimation of parameter $\boldsymbol{\beta}$ in each epoch is a base for calculation of parameter $\boldsymbol{\gamma}$ in the relations $y = \gamma_1 + \gamma_2 x + \gamma_3 x^2$ that describe trajectories of the reference points, e.g. in the case of the reference point P_1

$$\beta_{12}^{(i)} = \gamma_1 + \gamma_2 \beta_{11}^{(i)} + \gamma_3 (\beta_{11}^{(i)})^2, \quad i = 1, 2, 3, 4.$$

Estimation of parameters γ_1, γ_2 and γ_3 is executed by linearized regression model with constraint of type II because estimated coordinates $\hat{\boldsymbol{\beta}}^{(i)}$ are result of the measurement. Therefore the constraint is

$$\mathbf{BH} + \mathbf{CZ} + \mathbf{G} = \mathbf{0},$$

where $\mathbf{C}$ is a matrix of the parameter $\boldsymbol{\gamma}$ and

$$\mathbf{H} = \begin{pmatrix} \gamma_{12}^0 + 2\gamma_{11}^0 \beta_{11}^0 & -1 & 0 & 0 & \dots \\ 0 & 0 & \gamma_{22}^0 + 2\gamma_{21}^0 \beta_{21}^0 & -1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$\mathbf{Z} = \begin{pmatrix} 1 & \beta_{11}^0 & (\beta_{11}^0)^2 \\ 1 & \beta_{21}^0 & (\beta_{21}^0)^2 \\ \vdots & \vdots & \vdots \end{pmatrix},$$

$$\mathbf{G} = \begin{pmatrix} \gamma_{11}^0 + \gamma_{12}^0 \beta_{11}^0 + \gamma_{13}^0 (\beta_{11}^0)^2 - \beta_{12}^0 \\ \gamma_{21}^0 + \gamma_{22}^0 \beta_{21}^0 + \gamma_{23}^0 (\beta_{21}^0)^2 - \beta_{22}^0 \\ \vdots \end{pmatrix}.$$

4 Numerical example

In the experiment the distances of the reference points from the points

$$A = [\theta_{A1}, \theta_{A2}],\ B = [\theta_{B1}, \theta_{B2}],\ C = [\theta_{C1}, \theta_{C2}],\ D = [\theta_{D1}, \theta_{D2}]$$

and the angles between these points are measured. Approximate coordinates are

$$P_1 = [1300.0\,\mathrm{m}, 450.0\,\mathrm{m}],\ P_2 = [1271.4\,\mathrm{m}, 498.2\,\mathrm{m}],\ P_3 = [1250.0\,\mathrm{m}, 550.0\,\mathrm{m}].$$

$$\boldsymbol{\Sigma} = \sigma_s^2 \begin{pmatrix} \mathbf{I}_{4\times4} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{8\times8} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{4\times4} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}_{8\times8} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{4\times4} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}_{8\times8} \end{pmatrix} + \sigma_\omega^2 \begin{pmatrix} \mathbf{0}_{4\times4} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{8\times8} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0}_{4\times4} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{8\times8} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}_{4\times4} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{8\times8} \end{pmatrix},$$

$\sigma_s^2 = (0.01\,\mathrm{m})^2$ and $\sigma_\omega^2 = (\frac{3}{206265})^2$, where ω is an angle measured in radians.

The origin of the system of the coordinates is moved to the coordinates $[1200\,\mathrm{m}, 400\,\mathrm{m}]$.

The structural designer gives these trajectories:

$$\begin{aligned} -222172.44 &+ 4444.4444\boldsymbol{\beta}_{11} - 22.2222\boldsymbol{\beta}_{11}^2 - \boldsymbol{\beta}_{12} = 0,\\ 80.35555 &+ 0.25\boldsymbol{\beta}_{21} - \boldsymbol{\beta}_{22} = 0,\\ 55705.61 &- 2222.2222\boldsymbol{\beta}_{31} + 22.2222\boldsymbol{\beta}_{31}^2 - \boldsymbol{\beta}_{32} = 0. \end{aligned}$$

The corrected coordinates $\widehat{\widehat{\mathbf{B}}}$ given in meters from the model (4) are given in the following Table 1:

	1st epoch	2nd epoch	3th epoch	4th epoch
P1	[99.991,50.005]	[100.008,50.003]	[100.022,49.989]	[100.035,49.979]
P2	[71.413,98.206]	[71.431,98.214]	[71.444,98.223]	[71.464,98.233]
P3	[50.000,150.004]	[50.008,150.004]	[50.033,150.012]	[50.035,150.023]

Table 1

and

$$\begin{aligned} -383388.15 &+ 7668.548048\boldsymbol{\beta}_{11} - 38.341665\boldsymbol{\beta}_{11}^2 - \boldsymbol{\beta}_{12} = 0,\\ 60.29 &+ 0.530932\boldsymbol{\beta}_{21} - \boldsymbol{\beta}_{22} = 0,\\ 94104.42 &- 3757.486631\boldsymbol{\beta}_{31} + 37.567967\boldsymbol{\beta}_{31}^2 - \boldsymbol{\beta}_{32} = 0. \end{aligned}$$

Although in the model (4) the estimations of the parameter $\boldsymbol{\gamma}$ are different from the parameter given by the structural designer, the estimated trajectories are practically the same in the sector in which the movements of the reference point have been measured.

The figures 2–4 show the specific inconsistence between the theory and the numerical results. The corrected coordinates should lie exactly on the each trajectories. The inconsistence evident from the figures seems to be made by the linearization of the nonlinear model. An influence of the nonlinearity will be characterized by the bias

$$\mathbf{E}(\widehat{\delta\boldsymbol{\beta}}) - \delta\boldsymbol{\beta},$$

where $\mathbf{E}(\widehat{\delta\boldsymbol{\beta}})$ is calculated under the assumption that the nonlinear model is quadratized. The expression

$$\mathbf{E}(\widehat{\delta\boldsymbol{\beta}}) - \delta\boldsymbol{\beta}$$

can be obtained in our case from the formula in [1], p. 248, Corollary VI. 2.2.3.5. For the numerical demonstration we use the relations

$$\mathbf{E}(\widehat{\delta\boldsymbol{\beta}}) - \delta\boldsymbol{\beta} = \mathbf{P}^{\mathbf{C}}_{\mathbf{C}^{-1}\mathbf{H}'\mathbf{M_Z}}[\mathbf{C}^{-1}\mathbf{H}'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}' + \mathbf{Z}\mathbf{Z}')^{-1}]\delta\boldsymbol{\mu}\delta\gamma_2$$

for the point P_2 and

$$\begin{aligned}\mathbf{E}(\widehat{\delta\boldsymbol{\beta}}) - \delta\boldsymbol{\beta} &= \mathbf{P}^{\mathbf{C}}_{\mathbf{C}^{-1}\mathbf{H}'\mathbf{M_Z}}[\mathbf{C}^{-1}\mathbf{H}'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}' + \mathbf{Z}\mathbf{Z}')^{-1}] \\ &\times \left(\delta\gamma_2\delta\beta_{11}^{(i)} + \gamma_3^0(\delta\beta_{11}^{(i)})^2 + 2\delta\gamma_3\delta\beta_{11}^{(i)}\beta_{11}^{0(i)}\right)_{i=1}^4\end{aligned}$$

for the points P_1, P_3, where $\mathbf{C} = \mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X}$, $\delta\gamma_i = \sqrt{\operatorname{var}\gamma_i}$, $\delta\boldsymbol{\mu} = 4 \times 1$ matrix containing arbitrary combination of numbers $0, 1, -1$.

Numerical results verify the influence of the nonlinearity.

$$\mathbf{E}(\widehat{\delta\boldsymbol{\beta}}) - \delta\boldsymbol{\beta} = \begin{array}{r} -0.002 \\ 0.004 \\ -0.002 \\ 0.008 \\ 0.002 \\ -0.006 \\ 0.004 \\ -0.020 \end{array}$$

for the point P_2, where $\delta\boldsymbol{\mu} = [0; 1; 0; 1]$ and $\delta\gamma_2 = 0.053$.

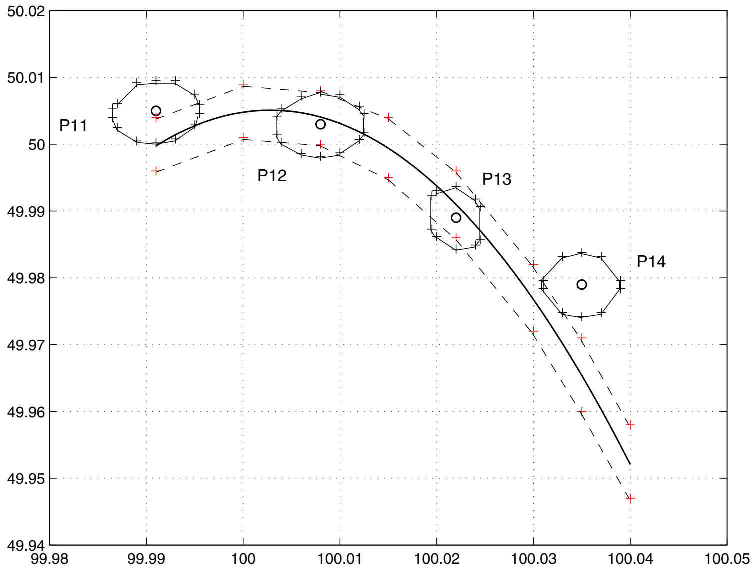

Fig. 2: P_1: Estimation of the trajectory + confidence region in the model (4).

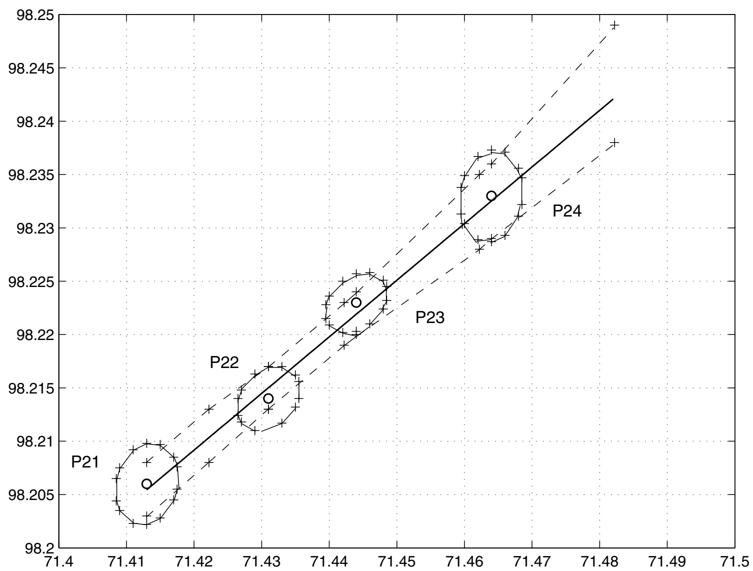

Fig. 3: P_2: Estimation of the trajectory + confidence region in the model (4).

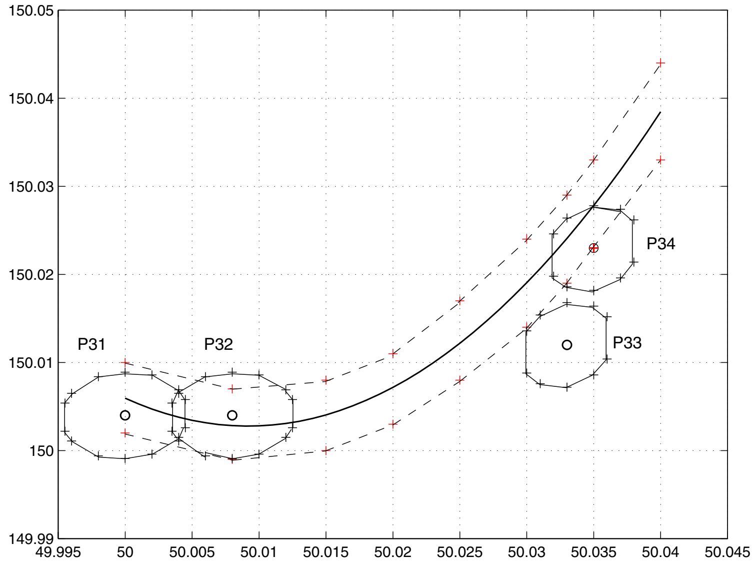

Fig. 4: P_3: Estimation of the trajectory + confidence region in the model (4).

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Inversion of 3 × 3 Partitioned Matrices in Investigation of the Twoepoch Linear Model with the Nuisance Parameters

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Abstract

The estimation procedures in the multiepoch (and specially twoepoch) linear regression models with the nuisance parameters that were described in [2], Chapter 9, frequently need finding the inverse of a 3×3 partitioned matrix. We use different kinds of such inversion in dependence on simplicity of the result, similarly as in well known Rohde formula for 2×2 partitioned matrix. We will show some of these formulas, also methods how to get the other formulas, and then we applicate the formulas in estimation of the mean value parameters in the twoepoch linear regression model with the nuisance parameters.

Key words: Inversion of partitioned matrices; Rohde formula; twoepoch regression model; useful and nuisance parameters; best linear estimators of the mean value parameter.

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1 Notations

The following notation will be used throughout the paper:

$\mathbb{R}^n$	the space of all n-dimensional real vectors;
$\mathbf{u}$, $\mathbf{A}$	the real column vector, the real matrix;
$\mathbf{A}', r(\mathbf{A})$	the transpose, the rank of the matrix $\mathbf{A}$;

$\mathcal{M}(\mathbf{A})$, $\mathrm{Ker}(\mathbf{A})$	the range, the null space of the matrix $\mathbf{A}$;
$\mathbf{A}^-$	a generalized inverse of a matrix $\mathbf{A}$ (satisfying $\mathbf{A}\mathbf{A}^-\mathbf{A} = \mathbf{A}$);
$\mathbf{A}^+$	the Moore-Penrose generalized inverse of a matrix $\mathbf{A}$ (satisfying $\mathbf{A}\mathbf{A}^+\mathbf{A} = \mathbf{A}, \mathbf{A}^+\mathbf{A}\mathbf{A}^+ = \mathbf{A}^+$, $(\mathbf{A}\mathbf{A}^+)' = \mathbf{A}\mathbf{A}^+, (\mathbf{A}^+\mathbf{A})' = \mathbf{A}^+\mathbf{A}$);
$\mathbf{P}_A$	the orthogonal projector onto $\mathcal{M}(\mathbf{A})$ (in Euclidean sense);
$\mathbf{M}_A = \mathbf{I} - \mathbf{P}_A$	the orthogonal projector onto $\mathcal{M}^\perp(\mathbf{A}) = \mathrm{Ker}(\mathbf{A}')$;
$\mathbf{I}_k$	the $k \times k$ identity matrix;
$\mathbf{0}_{m,n}$	the $m \times n$ null matrix;
$\mathbf{1}_k$	$= (1, \ldots, 1)' \in \mathbb{R}^k$;
χ^2_r	random variable with chi squared distribution with r degrees of freedom;
$\chi^2_r(1-\alpha)$	$(1-\alpha)$-quantile of this distribution.

If $\mathcal{M}(\mathbf{A}) \subset \mathcal{M}(\mathbf{S})$, $\mathbf{S}$ positive semidefinite (p.s.d.), then the symbol $\mathbf{P}_A^{S^-}$ denotes the projector projecting vectors in $\mathcal{M}(\mathbf{S})$ onto $\mathcal{M}(\mathbf{A})$ along $\mathcal{M}(\mathbf{S}\mathbf{A}^\perp)$. A general representation of all such projectors $\mathbf{P}_A^{S^-}$ is given by

$$\mathbf{A}(\mathbf{A}'\mathbf{S}^-\mathbf{A})^-\mathbf{A}'\mathbf{S}^- + \mathbf{B}(\mathbf{I} - \mathbf{S}\mathbf{S}^-),$$

where $\mathbf{B}$ is arbitrary, (see [4], (2.14)). $\mathbf{M}_A^{S^-} = \mathbf{I} - \mathbf{P}_A^{S^-}$.

2 Inversion of partitioned matrices

Lemma 1 (Rohde) *Let*

$$\mathbf{D} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}' & \mathbf{C} \end{pmatrix}$$

be (symmetric) positive definite (p.d.). Then

$$\mathbf{D}^{-1} =$$

$$= \begin{pmatrix} (\mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1} & -(\mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{C}^{-1} \\ -\mathbf{C}^{-1}\mathbf{B}'(\mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1} & \mathbf{C}^{-1} + \mathbf{C}^{-1}\mathbf{B}'(\mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{C}^{-1} \end{pmatrix} \quad (1)$$

$$= \begin{pmatrix} \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{B}(\mathbf{C} - \mathbf{B}'\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{B}'\mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{B}(\mathbf{C} - \mathbf{B}'\mathbf{A}^{-1}\mathbf{B})^{-1} \\ -(\mathbf{C} - \mathbf{B}'\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{B}'\mathbf{A}^{-1} & (\mathbf{C} - \mathbf{B}'\mathbf{A}^{-1}\mathbf{B})^{-1} \end{pmatrix}. \quad (2)$$

Proof see [1, Theorem 8.5.11, p. 99].

Theorem 1 (Version I) *Let*

$$\mathbf{Q} = \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{D} \\ \mathbf{B}' & \mathbf{C} & \mathbf{F} \\ \mathbf{D}' & \mathbf{F}' & \mathbf{E} \end{pmatrix}$$

be p.d. Then

$$\mathbf{Q}^{-1} = \begin{pmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} & \mathbf{Q}_{13} \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} & \mathbf{Q}_{23} \\ \mathbf{Q}_{31} & \mathbf{Q}_{32} & \mathbf{Q}_{33} \end{pmatrix},$$

where

$$\begin{aligned}
\mathbf{Q}_{11} &= [\mathbf{A} - \mathbf{D}\mathbf{E}^{-1}\mathbf{D}' - (\mathbf{B} - \mathbf{D}\mathbf{E}^{-1}\mathbf{F}')(\mathbf{C} - \mathbf{F}\mathbf{E}^{-1}\mathbf{F}')^{-1}(\mathbf{B}' - \mathbf{F}\mathbf{E}^{-1}\mathbf{D}')]^{-1}, \\
\mathbf{Q}_{12} &= -\mathbf{Q}_{11}(\mathbf{B} - \mathbf{D}\mathbf{E}^{-1}\mathbf{F}')(\mathbf{C} - \mathbf{F}\mathbf{E}^{-1}\mathbf{F}')^{-1}, \\
\mathbf{Q}_{13} &= -(\mathbf{Q}_{11}\mathbf{D} + \mathbf{Q}_{12}\mathbf{F})\mathbf{E}^{-1}, \\
\mathbf{Q}_{21} &= -(\mathbf{C} - \mathbf{F}\mathbf{E}^{-1}\mathbf{F}')^{-1}(\mathbf{B}' - \mathbf{F}\mathbf{E}^{-1}\mathbf{D}')\mathbf{Q}_{11} = (\mathbf{Q}_{12})', \\
\mathbf{Q}_{22} &= (\mathbf{C} - \mathbf{F}\mathbf{E}^{-1}\mathbf{F}')^{-1} + (\mathbf{C} - \mathbf{F}\mathbf{E}^{-1}\mathbf{F}')^{-1}(\mathbf{B}' - \mathbf{F}\mathbf{E}^{-1}\mathbf{D}')\mathbf{Q}_{11} \\
&\quad \times (\mathbf{B} - \mathbf{D}\mathbf{E}^{-1}\mathbf{F}')(\mathbf{C} - \mathbf{F}\mathbf{E}^{-1}\mathbf{F}')^{-1}, \\
\mathbf{Q}_{23} &= -(\mathbf{Q}_{21}\mathbf{D} + \mathbf{Q}_{22}\mathbf{F})\mathbf{E}^{-1}, \\
\mathbf{Q}_{31} &= -\mathbf{E}^{-1}(\mathbf{D}'\mathbf{Q}_{11} + \mathbf{F}'\mathbf{Q}_{21}) = (\mathbf{Q}_{13})', \\
\mathbf{Q}_{32} &= -\mathbf{E}^{-1}(\mathbf{D}'\mathbf{Q}_{12} + \mathbf{F}'\mathbf{Q}_{22}) = (\mathbf{Q}_{23})', \\
\mathbf{Q}_{33} &= \mathbf{E}^{-1} + \mathbf{E}^{-1}(\mathbf{D}'\mathbf{Q}_{11}\mathbf{D} + \mathbf{D}'\mathbf{Q}_{12}\mathbf{F} + \mathbf{F}'\mathbf{Q}_{21}\mathbf{D} + \mathbf{F}'\mathbf{Q}_{22}\mathbf{F})\mathbf{E}^{-1}.
\end{aligned}$$

Proof Let us denote

$$\mathbf{U} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}' & \mathbf{C} \end{pmatrix}, \qquad \mathbf{V} = \begin{pmatrix} \mathbf{D} \\ \mathbf{F} \end{pmatrix}.$$

The matrix $\mathbf{U}$ is p.d. so that we get with use of Lemma 1, formula (1)

$$\mathbf{Q}^{-1} = \begin{pmatrix} \mathbf{U} & \mathbf{V} \\ \mathbf{V}' & \mathbf{E} \end{pmatrix}^{-1}$$

$$\stackrel{(1)}{=} \begin{pmatrix} (\mathbf{U} - \mathbf{V}\mathbf{E}^{-1}\mathbf{V}')^{-1} & -(\mathbf{U} - \mathbf{V}\mathbf{E}^{-1}\mathbf{V}')^{-1}\mathbf{V}\mathbf{E}^{-1} \\ -\mathbf{E}^{-1}\mathbf{V}'(\mathbf{U} - \mathbf{V}\mathbf{E}^{-1}\mathbf{V}')^{-1} & \mathbf{E}^{-1} + \mathbf{E}^{-1}\mathbf{V}'(\mathbf{U} - \mathbf{V}\mathbf{E}^{-1}\mathbf{V}')^{-1}\mathbf{V}\mathbf{E}^{-1} \end{pmatrix}$$

with p.d. matrix

$$(\mathbf{U} - \mathbf{V}\mathbf{E}^{-1}\mathbf{V}')^{-1} = \begin{pmatrix} \mathbf{A} - \mathbf{D}\mathbf{E}^{-1}\mathbf{D}' & \mathbf{B} - \mathbf{D}\mathbf{E}^{-1}\mathbf{F}' \\ \mathbf{B}' - \mathbf{F}\mathbf{E}^{-1}\mathbf{D}' & \mathbf{C} - \mathbf{F}\mathbf{E}^{-1}\mathbf{F}' \end{pmatrix}^{-1}.$$

An application of Rohde formula (1) again and arrangement give us the desired result. □

Corollary 1 *Inverse of partitioned p.d. matrix*

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{D} \\ \mathbf{B}' & \mathbf{C} & \mathbf{0} \\ \mathbf{D}' & \mathbf{0} & \mathbf{E} \end{pmatrix}$$

is equal to

$$\begin{pmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} & \mathbf{Q}_{13} \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} & \mathbf{Q}_{23} \\ \mathbf{Q}_{31} & \mathbf{Q}_{32} & \mathbf{Q}_{33} \end{pmatrix} =$$

$$= \begin{pmatrix} \mathbf{Q}_{11} & -\mathbf{Q}_{11}\mathbf{B}\mathbf{C}^{-1} & -\mathbf{Q}_{11}\mathbf{D}\mathbf{E}^{-1} \\ -\mathbf{C}^{-1}\mathbf{B}'\mathbf{Q}_{11} & \mathbf{C}^{-1}+\mathbf{C}^{-1}\mathbf{B}'\mathbf{Q}_{11}\mathbf{B}\mathbf{C}^{-1} & -\mathbf{Q}_{21}\mathbf{D}\mathbf{E}^{-1} \\ -\mathbf{E}^{-1}\mathbf{D}'\mathbf{Q}_{11} & -\mathbf{E}^{-1}\mathbf{D}'\mathbf{Q}_{12} & \mathbf{E}^{-1}+\mathbf{E}^{-1}\mathbf{D}'\mathbf{Q}_{11}\mathbf{D}\mathbf{E}^{-1} \end{pmatrix},$$

where

$$\mathbf{Q}_{11} = (\mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}' - \mathbf{D}\mathbf{E}^{-1}\mathbf{D}')^{-1}.$$

Theorem 2 (Version II) *Let*

$$\mathbf{Q} = \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{D} \\ \mathbf{B}' & \mathbf{C} & \mathbf{F} \\ \mathbf{D}' & \mathbf{F}' & \mathbf{E} \end{pmatrix}$$

be p.d. Then

$$\mathbf{Q}^{-1} = \begin{pmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} & \mathbf{Q}_{13} \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} & \mathbf{Q}_{23} \\ \mathbf{Q}_{31} & \mathbf{Q}_{32} & \mathbf{Q}_{33} \end{pmatrix},$$

where

$$\begin{aligned}
\mathbf{Q}_{11} &= (\mathbf{A} - \mathbf{D}\mathbf{E}^{-1}\mathbf{D}')^{-1} + (\mathbf{A} - \mathbf{D}\mathbf{E}^{-1}\mathbf{D}')^{-1}(\mathbf{B} - \mathbf{D}\mathbf{E}^{-1}\mathbf{F}')\mathbf{Q}_{22} \\
&\quad \times (\mathbf{B}' - \mathbf{F}\mathbf{E}^{-1}\mathbf{D}')(\mathbf{A} - \mathbf{D}\mathbf{E}^{-1}\mathbf{D}')^{-1}, \\
\mathbf{Q}_{12} &= -(\mathbf{A} - \mathbf{D}\mathbf{E}^{-1}\mathbf{D}')^{-1}(\mathbf{B} - \mathbf{D}\mathbf{E}^{-1}\mathbf{F}')\mathbf{Q}_{22}, \\
\mathbf{Q}_{13} &= -(\mathbf{Q}_{11}\mathbf{D} + \mathbf{Q}_{12}\mathbf{F})\mathbf{E}^{-1}, \\
\mathbf{Q}_{21} &= -\mathbf{Q}_{22}(\mathbf{B}' - \mathbf{F}\mathbf{E}^{-1}\mathbf{D}')(\mathbf{A} - \mathbf{D}\mathbf{E}^{-1}\mathbf{D}')^{-1}, \\
\mathbf{Q}_{22} &= [\mathbf{C} - \mathbf{F}\mathbf{E}^{-1}\mathbf{F}' - (\mathbf{B}' - \mathbf{F}\mathbf{E}^{-1}\mathbf{D}')(\mathbf{A} - \mathbf{D}\mathbf{E}^{-1}\mathbf{D}')^{-1}(\mathbf{B} - \mathbf{D}\mathbf{E}^{-1}\mathbf{F}')]^{-1}, \\
\mathbf{Q}_{23} &= -(\mathbf{Q}_{21}\mathbf{D} + \mathbf{Q}_{22}\mathbf{F})\mathbf{E}^{-1}, \\
\mathbf{Q}_{31} &= -\mathbf{E}^{-1}(\mathbf{D}'\mathbf{Q}_{11} + \mathbf{F}'\mathbf{Q}_{21}), \\
\mathbf{Q}_{32} &= -\mathbf{E}^{-1}(\mathbf{D}'\mathbf{Q}_{12} + \mathbf{F}'\mathbf{Q}_{22}), \\
\mathbf{Q}_{33} &= \mathbf{E}^{-1} + \mathbf{E}^{-1}(\mathbf{D}'\mathbf{Q}_{11}\mathbf{D} + \mathbf{D}'\mathbf{Q}_{12}\mathbf{F} + \mathbf{F}'\mathbf{Q}_{21}\mathbf{D} + \mathbf{F}'\mathbf{Q}_{22}\mathbf{F})\mathbf{E}^{-1}.
\end{aligned}$$

Proof follows directly from the proof of Theorem 1, if we use Rohde formula (2) instead of (1) in inverting p.d. matrix

$$\begin{pmatrix} \mathbf{A} - \mathbf{D}\mathbf{E}^{-1}\mathbf{D}' & \mathbf{B} - \mathbf{D}\mathbf{E}^{-1}\mathbf{F}' \\ \mathbf{B}' - \mathbf{F}\mathbf{E}^{-1}\mathbf{D}' & \mathbf{C} - \mathbf{F}\mathbf{E}^{-1}\mathbf{F}' \end{pmatrix}^{-1}. \qquad \square$$

Remark 1 (Version III & Version IV) We use (1) and (2) in inverting p.d. matrix

$$\mathbf{U} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}' & \mathbf{C} \end{pmatrix}$$

in

$$\mathbf{Q}^{-1} = \begin{pmatrix} \mathbf{U} & \mathbf{V} \\ \mathbf{V}' & \mathbf{E} \end{pmatrix}^{-1}$$

$$\overset{(2)}{=} \begin{pmatrix} \mathbf{U}^{-1} + \mathbf{U}^{-1}\mathbf{V}(\mathbf{E} - \mathbf{V}'\mathbf{U}^{-1}\mathbf{V})^{-1}\mathbf{V}'\mathbf{U}^{-1} & -\mathbf{U}^{-1}\mathbf{V}(\mathbf{E} - \mathbf{V}'\mathbf{U}^{-1}\mathbf{V})^{-1} \\ -(\mathbf{E} - \mathbf{V}'\mathbf{U}^{-1}\mathbf{V})^{-1}\mathbf{V}'\mathbf{U}^{-1} & (\mathbf{E} - \mathbf{V}'\mathbf{U}^{-1}\mathbf{V})^{-1} \end{pmatrix},$$

where $\mathbf{V} = (\mathbf{D}', \mathbf{F}')'$.

Remark 2 (Version V & Version VI) Let us denote

$$\mathbf{W} = (\mathbf{B}, \mathbf{D}), \qquad \mathbf{Z} = \begin{pmatrix} \mathbf{C} & \mathbf{F} \\ \mathbf{F}' & \mathbf{E} \end{pmatrix}$$

in

$$\mathbf{Q} = \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{D} \\ \mathbf{B}' & \mathbf{C} & \mathbf{F} \\ \mathbf{D}' & \mathbf{F}' & \mathbf{E} \end{pmatrix}.$$

The matrix $\mathbf{Z}$ is p.d. and using (1) we get

$$\mathbf{Q}^{-1} = \begin{pmatrix} \mathbf{A} & \mathbf{W} \\ \mathbf{W}' & \mathbf{Z} \end{pmatrix}^{-1}$$

$$\overset{(1)}{=} \begin{pmatrix} (\mathbf{A} - \mathbf{W}\mathbf{Z}^{-1}\mathbf{W}')^{-1} & -(\mathbf{A} - \mathbf{W}\mathbf{Z}^{-1}\mathbf{W}')^{-1}\mathbf{W}\mathbf{Z}^{-1} \\ -\mathbf{Z}^{-1}\mathbf{W}'(\mathbf{A} - \mathbf{W}\mathbf{Z}^{-1}\mathbf{W}')^{-1} & \mathbf{Z}^{-1} + \mathbf{Z}^{-1}\mathbf{W}'(\mathbf{A} - \mathbf{W}\mathbf{Z}^{-1}\mathbf{W}')^{-1}\mathbf{W}\mathbf{Z}^{-1} \end{pmatrix}.$$

The only thing that remains is to invert $\mathbf{Z}$ by (1) and (2).

Remark 3 (Version VII & Version VIII) Using Rohde formula (2) in p.d. matrix inversion

$$\mathbf{Q}^{-1} = \begin{pmatrix} \mathbf{A} & \mathbf{W} \\ \mathbf{W}' & \mathbf{Z} \end{pmatrix}^{-1}$$

we obtain

$$\begin{pmatrix} \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{W}(\mathbf{Z} - \mathbf{W}'\mathbf{A}^{-1}\mathbf{W})^{-1}\mathbf{W}'\mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{W}(\mathbf{Z} - \mathbf{W}'\mathbf{A}^{-1}\mathbf{W})^{-1} \\ -(\mathbf{Z} - \mathbf{W}'\mathbf{A}^{-1}\mathbf{W})^{-1}\mathbf{W}'\mathbf{A}^{-1} & (\mathbf{Z} - \mathbf{W}'\mathbf{A}^{-1}\mathbf{W})^{-1} \end{pmatrix}$$

with p.d. matrix

$$(\mathbf{Z} - \mathbf{W}'\mathbf{A}^{-1}\mathbf{W})^{-1} = \begin{pmatrix} \mathbf{C} - \mathbf{B}'\mathbf{A}^{-1}\mathbf{B} & \mathbf{F} - \mathbf{B}'\mathbf{A}^{-1}\mathbf{D} \\ \mathbf{F}' - \mathbf{D}'\mathbf{A}^{-1}\mathbf{B} & \mathbf{E} - \mathbf{D}'\mathbf{A}^{-1}\mathbf{D} \end{pmatrix}^{-1}.$$

An application of (1) and (2) again give us the result. For

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{D} \\ \mathbf{B}' & \mathbf{C} & \mathbf{F} \\ \mathbf{D}' & \mathbf{F}' & \mathbf{E} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} & \mathbf{Q}_{13} \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} & \mathbf{Q}_{23} \\ \mathbf{Q}_{31} & \mathbf{Q}_{32} & \mathbf{Q}_{33} \end{pmatrix},$$

it is interesting to compare Version VIII,

$$\begin{aligned}
\mathbf{Q}_{11} &= \mathbf{A}^{-1} + \mathbf{A}^{-1}(\mathbf{B}\mathbf{Q}_{22}\mathbf{B}' + \mathbf{B}\mathbf{Q}_{23}\mathbf{D}' + \mathbf{D}\mathbf{Q}_{32}\mathbf{B}' + \mathbf{D}\mathbf{Q}_{33}\mathbf{D}')\mathbf{A}^{-1}, \\
\mathbf{Q}_{12} &= -\mathbf{A}^{-1}(\mathbf{B}\mathbf{Q}_{22} + \mathbf{D}\mathbf{Q}_{32}), \\
\mathbf{Q}_{13} &= -\mathbf{A}^{-1}(\mathbf{B}\mathbf{Q}_{23} + \mathbf{D}\mathbf{Q}_{33}), \\
\mathbf{Q}_{21} &= -(\mathbf{Q}_{22}\mathbf{B}' + \mathbf{Q}_{23}\mathbf{D}')\mathbf{A}^{-1}, \\
\mathbf{Q}_{22} &= (\mathbf{C} - \mathbf{B}'\mathbf{A}^{-1}\mathbf{B})^{-1} + (\mathbf{C} - \mathbf{B}'\mathbf{A}^{-1}\mathbf{B})^{-1}(\mathbf{F} - \mathbf{B}'\mathbf{A}^{-1}\mathbf{D})\mathbf{Q}_{33} \\
&\quad \times (\mathbf{F}' - \mathbf{D}'\mathbf{A}^{-1}\mathbf{B})(\mathbf{C} - \mathbf{B}'\mathbf{A}^{-1}\mathbf{B})^{-1}, \\
\mathbf{Q}_{23} &= -(\mathbf{C} - \mathbf{B}'\mathbf{A}^{-1}\mathbf{B})^{-1}(\mathbf{F} - \mathbf{B}'\mathbf{A}^{-1}\mathbf{D})\mathbf{Q}_{33}, \\
\mathbf{Q}_{31} &= -(\mathbf{Q}_{32}\mathbf{B}' + \mathbf{Q}_{33}\mathbf{D}')\mathbf{A}^{-1}, \\
\mathbf{Q}_{32} &= -\mathbf{Q}_{33}(\mathbf{F}' - \mathbf{D}'\mathbf{A}^{-1}\mathbf{B})(\mathbf{C} - \mathbf{B}'\mathbf{A}^{-1}\mathbf{B})^{-1}, \\
\mathbf{Q}_{33} &= [\mathbf{E} - \mathbf{D}'\mathbf{A}^{-1}\mathbf{D} - (\mathbf{F}' - \mathbf{D}'\mathbf{A}^{-1}\mathbf{B})(\mathbf{C} - \mathbf{B}'\mathbf{A}^{-1}\mathbf{B})^{-1}(\mathbf{F} - \mathbf{B}'\mathbf{A}^{-1}\mathbf{D})]^{-1},
\end{aligned}$$

with Version I—it's in the certain sense "dual" form of Version VIII. Similar comparisons can be done with other couples of formulas.

3 Twoepoch linear model

The theory of the linear regression models is one of the established statistical disciplines and it may seem that nearly all has been investigated there. But this is valid only for the simplest structures of the linear models. In the practice we need to solve more and more complicated problems and investigation of corresponding structures of models is at the beginning. The formulas are quite complicated there but easy programmable and it enables us to get the estimations of unknown parameters in linear models.

The estimation procedures in multiepoch linear regression models with nuisance parameters and its application in geodesy were described in [2, Chapter 9]. But in the twoepoch case we can derive the estimations using convenient inverse of 3×3 partitioned matrices much easily so it legitimates to deal with them specially.

We derive optimum estimators of the useful mean value within a linear twoepoch model with the stable and variable (nonstable) parameters, when the data are affected by a systematic (deterministic) influence, i.e. by a noise which can be described by a linear model and whose parameters called nuisance, are estimable from results of the measurement. The subject of an interpretation are changes of the useful parameters in the single epochs and their characteristics of accuracy.

Sometimes the dimension of the useful mean value parameters is essentially smaller than that one of the nuisance parameter. In connection with this fact the problem occurs how to determine the optimum estimators of the useful parameters and their accuracy without evaluating in each epoch the large vector of the nuisance parameters.

One of the fundamental types of multiepoch and specially twoepoch model (which may exist also in the form with the nuisance parameters) was described in [2, p. 366].

Replicated measurements studying existence of deformation of some object and its course (if it exists) are realized in separate networks especially constructed for this purpose. It consists of a group of supporting points, whose position is assumed to be stable (this assumption—hypothesis—is verified during the measurement), and a group of points, whose movements related to the position of the stable points, are investigated (the coordinates of the group of the stable points are a priori unknown). As far as the processing of the measured results is concerned this means, that in the framework of each epoch and after finishing each epoch both the coordinates of the supporting points and the coordinates of the investigated points, are to be determined. The former serve to verify the above-mentioned hypothesis on the stableness of the group of supporting points.

Let us describe another example from the microeconomics practice. The progress of daily receipts in retail trade in the same months of two following years is observed. This progress usually consists of weekly period part and trend part. The weekly period doesn't change a lot because of conservative behaviour of the shoppers (i.e. useful stable parameters in expression of the entire linear model modelling the situation) in contrast to the trend. There is an influence of the commercial offers, inflation etc. (i.e. variable parameters; we suppose that the annual changes are not dramatical). The trend can be quite complicated and we need often only a small fraction of information that it contains. Here, the nonstable parameters in case of quadratic trend can be divided into the useful linear term parameter, that gives some pieces of information about increase or decrease of receipts, and two nuisance parameters (absolute term and quadratic term). The data in the above mentioned problem are usually characterized by a large dispersion and dependence among them.

The result of the measurement at the i-th time point in the first epoch could be described as

$$Y_{1i} = \beta_1 \cos \lambda t_{1i} + \beta_2 \sin \lambda t_{1i} + \gamma_1 t_{1i} + \kappa_{11} + \kappa_{12} t_{1i}^2 + \varepsilon_{1i}, \quad i = 1, \dots, n_1$$

(λ is known from periodogram, see [5, p. 92]) and

$$Y_{2i} = \beta_1 \cos \lambda t_{2i} + \beta_2 \sin \lambda t_{2i} + \gamma_2 t_{2i} + \kappa_{21} + \kappa_{22} t_{2i}^2 + \varepsilon_{2i}, \quad i = 1, \dots, n_2$$

in the second epoch. Here $\beta_1 \cos \lambda t_{ji} + \beta_2 \sin \lambda t_{ji}$ describes the weekly period (the measurements must begin with respect to this period in both epochs) and $\gamma_j t_{ji} + \kappa_{j1} + \kappa_{j2} t_{ji}^2$, $j = 1, 2$ the quadratical trend in the first and second epoch, respectively.

Let us consider the observation vector $\mathbf{Y} = (\mathbf{Y}_1', \mathbf{Y}_2')'$. The model described above could be rewritten in the form

$$\begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{X}_1 & \mathbf{W}_1 & \mathbf{0} & \mathbf{Z}_1 & \mathbf{0} \\ \mathbf{X}_2 & \mathbf{0} & \mathbf{W}_2 & \mathbf{0} & \mathbf{Z}_2 \end{pmatrix} \begin{pmatrix} \boldsymbol{\beta} \\ \boldsymbol{\gamma} \\ \boldsymbol{\kappa} \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}, \tag{3}$$

where

$$\mathbf{X}_1 = \begin{pmatrix} \cos \lambda t_{11} & \sin \lambda t_{11} \\ \vdots & \vdots \\ \cos \lambda t_{1n_1} & \sin \lambda t_{1n_1} \end{pmatrix}, \qquad \mathbf{X}_2 = \begin{pmatrix} \cos \lambda t_{21} & \sin \lambda t_{21} \\ \vdots & \vdots \\ \cos \lambda t_{2n_2} & \sin \lambda t_{2n_2} \end{pmatrix},$$

$$\mathbf{W}_1 = (t_{11}, \dots, t_{1n_1})', \qquad \mathbf{W}_2 = (t_{21}, \dots, t_{2n_2})',$$

$$\mathbf{Z}_1 = \begin{pmatrix} 1 & t_{11}^2 \\ \vdots & \vdots \\ 1 & t_{1n_1}^2 \end{pmatrix}, \qquad \mathbf{Z}_2 = \begin{pmatrix} 1 & t_{21}^2 \\ \vdots & \vdots \\ 1 & t_{2n_2}^2 \end{pmatrix},$$

$$\boldsymbol{\beta} = (\beta_1, \beta_2)', \qquad \boldsymbol{\gamma} = (\gamma_1, \gamma_2)', \qquad \boldsymbol{\kappa} = (\kappa_{11}, \kappa_{12}, \kappa_{21}, \kappa_{22})'.$$

The matrices $\mathbf{X}_1, \mathbf{X}_2, \mathbf{W}_1, \mathbf{W}_2, \mathbf{Z}_1, \mathbf{Z}_2$ are known, the vector $\boldsymbol{\beta}$ is a vector of the useful stable parameters, $\boldsymbol{\gamma}$ is a vector of the useful variable parameters and $\boldsymbol{\kappa}$ is a vector of the nuisance variable parameters.

With respect to above mentioned, let us consider the linear model (3), called the twoepoch model with the stable and nonstable parameters and with the nuisance parameters. We suppose that

- $(\mathbf{Y}_1', \mathbf{Y}_2')'$ is a $(n_1 + n_2)$-dimensional random observation vector after the second epoch of measurement,
- $\boldsymbol{\beta} \in \mathbb{R}^k$ is a vector of the useful stable parameters, the same in both epochs,
- $\boldsymbol{\gamma} = (\boldsymbol{\gamma}_1', \boldsymbol{\gamma}_2')' \in \mathbb{R}^{l_1+l_2}$ is a vector of the useful nonstable parameters in the first and the second epoch of measurement,
- $\boldsymbol{\kappa} = (\boldsymbol{\kappa}_1', \boldsymbol{\kappa}_2')' \in \mathbb{R}^{s_1+s_2}$ is a vector of the nuisance nonstable parameters in first and second epoch,
- $\mathbf{X}_1$, $\mathbf{X}_2$ are $n_1 \times k$, $n_2 \times k$ design matrices belonging to the vector $\boldsymbol{\beta}$,
- $\mathbf{W}_1$ is a $n_1 \times l_1$ design matrix belonging to the vector $\boldsymbol{\gamma}_1$,
- $\mathbf{W}_2$ is a $n_2 \times l_2$ design matrix belonging to the vector $\boldsymbol{\gamma}_2$,
- $\mathbf{Z}_1$ is a $n_1 \times s_1$ design matrix belonging to the vector $\boldsymbol{\kappa}_1$,
- $\mathbf{Z}_2$ is a $n_2 \times s_2$ design matrix belonging to the vector $\boldsymbol{\kappa}_2$.

We suppose that

1. $\mathrm{E}(\mathbf{Y}_1) = \mathbf{X}_1\boldsymbol{\beta} + \mathbf{W}_1\boldsymbol{\gamma}_1 + \mathbf{Z}_1\boldsymbol{\kappa}_1$, $\mathrm{E}(\mathbf{Y}_2) = \mathbf{X}_2\boldsymbol{\beta} + \mathbf{W}_2\boldsymbol{\gamma}_2 + \mathbf{Z}_2\boldsymbol{\kappa}_2$, $\forall\boldsymbol{\beta} \in \mathbb{R}^k, \forall\boldsymbol{\gamma}_1 \in \mathbb{R}^{l_1}, \forall\boldsymbol{\gamma}_2 \in \mathbb{R}^{l_2}, \forall\boldsymbol{\kappa}_1 \in \mathbb{R}^{s_1}, \forall\boldsymbol{\kappa}_2 \in \mathbb{R}^{s_2}$;
2. $\mathrm{var}\left[\begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix}\right] = \begin{pmatrix} \boldsymbol{\Sigma}_1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_2 \end{pmatrix}$,
3. the matrix $\boldsymbol{\Sigma}_i$ is not a function of the vector $(\boldsymbol{\beta}', \boldsymbol{\gamma}_i', \boldsymbol{\kappa}_i')'$ for $i = 1, 2$.

If the matrix $\left(\begin{smallmatrix} \boldsymbol{\Sigma}_1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_2 \end{smallmatrix}\right)$ is p.d. and

$$r\left[\begin{pmatrix} \mathbf{X}_1 & \mathbf{W}_1 & \mathbf{0} & \mathbf{Z}_1 & \mathbf{0} \\ \mathbf{X}_2 & \mathbf{0} & \mathbf{W}_2 & \mathbf{0} & \mathbf{Z}_2 \end{pmatrix}\right] = k + l_1 + l_2 + s_1 + s_2 < n_1 + n_2,$$

the model is said to be *regular* (see [2, p. 13]).

The described model arises by sequential realizations of the linear partial regression models,

$$\mathbf{Y}_1 = (\mathbf{X}_1, \mathbf{W}_1, \mathbf{Z}_1)\begin{pmatrix} \boldsymbol{\beta} \\ \boldsymbol{\gamma}_1 \\ \boldsymbol{\kappa}_1 \end{pmatrix} + \boldsymbol{\varepsilon}_1, \quad \mathrm{var}(\mathbf{Y}_1) = \boldsymbol{\Sigma}_1 \tag{4}$$

and

$$\mathbf{Y}_2 = (\mathbf{X}_2, \mathbf{W}_2, \mathbf{Z}_2)\begin{pmatrix}\boldsymbol{\beta}\\ \gamma_2\\ \boldsymbol{\kappa}_2\end{pmatrix} + \boldsymbol{\varepsilon}_2, \quad \mathrm{var}(\mathbf{Y}_2) = \boldsymbol{\Sigma}_2, \tag{5}$$

representing the model of the measurement within the first and second epoch, respectively.

Theorem 3 *The BLUE, i.e. the best linear unbiased estimator, of the parameters* $\boldsymbol{\beta}, \gamma_i, \boldsymbol{\kappa}_i$, $i = 1, 2$ *in the single first and second epoch modelled by (4) and (5), respectively, are*

$$\begin{aligned}
\widehat{\boldsymbol{\beta}}^{(i)} &= (\mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{M}_{W_i}^{\Sigma_i^{-1}M_{Z_i}^{\Sigma_i^{-1}}}\mathbf{X}_i)^{-1}\mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{M}_{W_i}^{\Sigma_i^{-1}M_{Z_i}^{\Sigma_i^{-1}}}\mathbf{Y}_i,\\
\widehat{\gamma_i}^{(i)} &= (\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i)^{-1}\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}(\mathbf{Y}_i - \mathbf{X}_i\widehat{\boldsymbol{\beta}}^{(i)}),\\
\widehat{\boldsymbol{\kappa}_i}^{(i)} &= (\mathbf{Z}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{Z}_i)^{-1}\mathbf{Z}_i'\boldsymbol{\Sigma}_i^{-1}(\mathbf{Y}_i - \mathbf{X}_i\widehat{\boldsymbol{\beta}}^{(i)} - \mathbf{W}_i\widehat{\gamma}_i^{(i)}),
\end{aligned}$$

(Version I) and equivalently

$$\begin{aligned}
\widehat{\boldsymbol{\beta}}^{(i)} &= (\mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i)^{-1}\mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}(\mathbf{Y}_i - \mathbf{W}_i\widehat{\gamma}_i^{(i)}),\\
\widehat{\gamma_i}^{(i)} &= (\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{M}_{X_i}^{\Sigma_i^{-1}M_{Z_i}^{\Sigma_i^{-1}}}\mathbf{W}_i)^{-1}\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{M}_{X_i}^{\Sigma_i^{-1}M_{Z_i}^{\Sigma_i^{-1}}}\mathbf{Y}_i,\\
\widehat{\boldsymbol{\kappa}_i}^{(i)} &= (\mathbf{Z}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{Z}_i)^{-1}\mathbf{Z}_i'\boldsymbol{\Sigma}_i^{-1}(\mathbf{Y}_i - \mathbf{X}_i\widehat{\boldsymbol{\beta}}^{(i)} - \mathbf{W}_i\widehat{\gamma}_i^{(i)}),
\end{aligned}$$

(Version II) for $i = 1, 2$.

Proof According to [2, Theorem 1.1.1, p. 13], the BLUE of the vector parameter $(\boldsymbol{\beta}', \gamma_i', \boldsymbol{\kappa}_i')'$, $i = 1, 2$, in each epoch separately, is given by

$$\begin{pmatrix}\widehat{\boldsymbol{\beta}}^{(i)}\\ \widehat{\gamma_i}^{(i)}\\ \widehat{\boldsymbol{\kappa}_i}^{(i)}\end{pmatrix} = \left[\begin{pmatrix}\mathbf{X}_i'\\ \mathbf{W}_i'\\ \mathbf{Z}_i'\end{pmatrix}\boldsymbol{\Sigma}_i^{-1}(\mathbf{X}_i, \mathbf{W}_i, \mathbf{Z}_i)\right]^{-1}\begin{pmatrix}\mathbf{X}_i'\\ \mathbf{W}_i'\\ \mathbf{Z}_i'\end{pmatrix}\boldsymbol{\Sigma}_i^{-1}\mathbf{Y}_i.$$

Using Theorem 1 and Theorem 2, the crucial point of the proof consists in the fact that

$$\left[\begin{pmatrix}\mathbf{X}_i'\\ \mathbf{W}_i'\\ \mathbf{Z}_i'\end{pmatrix}\boldsymbol{\Sigma}_i^{-1}(\mathbf{X}_i, \mathbf{W}_i, \mathbf{Z}_i)\right]^{-1} =$$

$$= \begin{pmatrix}\mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{X}_i & \mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{W}_i & \mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{Z}_i\\ \mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{X}_i & \mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{W}_i & \mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{Z}_i\\ \mathbf{Z}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{X}_i & \mathbf{Z}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{W}_i & \mathbf{Z}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{Z}_i\end{pmatrix}^{-1} = \begin{pmatrix}\mathbf{Q}_{11} & \mathbf{Q}_{12} & \mathbf{Q}_{13}\\ \mathbf{Q}_{21} & \mathbf{Q}_{22} & \mathbf{Q}_{23}\\ \mathbf{Q}_{31} & \mathbf{Q}_{32} & \mathbf{Q}_{33}\end{pmatrix},$$

where $(\mathbf{M}_{Z_i}^{\Sigma_i^{-1}} = \mathbf{I} - \mathbf{P}_{Z_i}^{\Sigma_i^{-1}} = \mathbf{I} - \mathbf{Z}_i(\mathbf{Z}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{Z}_i)^{-1}\mathbf{Z}_i'\boldsymbol{\Sigma}_i^{-1})$

$$
\begin{aligned}
\mathbf{Q}_{11} &\overset{\text{T.1}}{=} [\mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i - \mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i(\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i)^{-1} \\
&\quad \times \mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i]^{-1} = (\mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{M}_{W_i}^{\Sigma_i^{-1}M_{Z_i}^{\Sigma_i^{-1}}}\mathbf{X}_i)^{-1}, \\
\mathbf{Q}_{12} &= -\mathbf{Q}_{11}\mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i(\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i)^{-1}, \\
\mathbf{Q}_{13} &= -\mathbf{Q}_{11}[\mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{Z}_i - \mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i(\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i)^{-1}\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{Z}_i] \\
&\quad \times (\mathbf{Z}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{Z}_i)^{-1}, \\
\mathbf{Q}_{21} &= -(\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i)^{-1}\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i\mathbf{Q}_{11}, \\
\mathbf{Q}_{22} &= (\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i)^{-1} + (\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i)^{-1}\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i \\
&\quad \times \mathbf{Q}_{11}\mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i(\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i)^{-1}, \\
\mathbf{Q}_{23} &= -(\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i)^{-1}[\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{Z}_i - \mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i\mathbf{Q}_{11}\mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{Z}_i \\
&\quad + \mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i\mathbf{Q}_{11}\mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i(\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i)^{-1} \\
&\quad \times \mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{Z}_i](\mathbf{Z}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{Z}_i)^{-1}, \\
\mathbf{Q}_{31} &= -(\mathbf{Z}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{Z}_i)^{-1}[\mathbf{Z}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{X}_i - \mathbf{Z}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{W}_i(\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i)^{-1} \\
&\quad \times \mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i]\mathbf{Q}_{11}, \\
\mathbf{Q}_{32} &= -(\mathbf{Z}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{Z}_i)^{-1}[\mathbf{Z}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{W}_i - \mathbf{Z}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{X}_i\mathbf{Q}_{11}\mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i \\
&\quad + \mathbf{Z}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{W}_i(\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i)^{-1}\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}} \\
&\quad \times \mathbf{X}_i\mathbf{Q}_{11}\mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i](\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i)^{-1}, \\
\mathbf{Q}_{33} &= (\mathbf{Z}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{Z}_i)^{-1} + (\mathbf{Z}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{Z}_i)^{-1}[\mathbf{Z}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{X}_i\mathbf{Q}_{11}\mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{Z}_i \\
&\quad - \mathbf{Z}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{X}_i\mathbf{Q}_{11}\mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i(\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i)^{-1}\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{Z}_i \\
&\quad - \mathbf{Z}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{W}_i(\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i)^{-1}\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i\mathbf{Q}_{11}\mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{Z}_i \\
&\quad + \mathbf{Z}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{W}_i(\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i)^{-1}\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{Z}_i \\
&\quad + \mathbf{Z}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{W}_i(\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i)^{-1}\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i\mathbf{Q}_{11}\mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i \\
&\quad \times (\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i)^{-1}\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{Z}_i](\mathbf{Z}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{Z}_i)^{-1}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{Q}_{11} &\overset{\text{T.2}}{=} (\mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i)^{-1} + (\mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i)^{-1}\mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i\mathbf{Q}_{22} \\
&\quad \times \mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i(\mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i)^{-1}, \\
\mathbf{Q}_{12} &= -(\mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i)^{-1}\mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i\mathbf{Q}_{22},
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{Q}_{13} &= -(\mathbf{X}_i'\mathbf{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i)^{-1}[\mathbf{X}_i'\mathbf{\Sigma}_i^{-1}\mathbf{Z}_i - \mathbf{X}_i'\mathbf{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i\mathbf{Q}_{22}\mathbf{W}_i'\mathbf{\Sigma}_i^{-1}\mathbf{Z}_i \\
&\quad + \mathbf{X}_i'\mathbf{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i\mathbf{Q}_{22}\mathbf{W}_i'\mathbf{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i(\mathbf{X}_i'\mathbf{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i)^{-1} \\
&\quad \times \mathbf{X}_i'\mathbf{\Sigma}_i^{-1}\mathbf{Z}_i](\mathbf{Z}_i'\mathbf{\Sigma}_i^{-1}\mathbf{Z}_i)^{-1}, \\
\mathbf{Q}_{21} &= -\mathbf{Q}_{22}\mathbf{W}_i'\mathbf{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i(\mathbf{X}_i'\mathbf{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i)^{-1}, \\
\mathbf{Q}_{22} &= (\mathbf{W}_i'\mathbf{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{M}_{X_i}^{\Sigma_i^{-1}M_{Z_i}^{\Sigma_i^{-1}}}\mathbf{W}_i)^{-1}, \\
\mathbf{Q}_{23} &= -\mathbf{Q}_{22}[\mathbf{W}_i'\mathbf{\Sigma}_i^{-1}\mathbf{Z}_i - \mathbf{W}_i'\mathbf{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i(\mathbf{X}_i'\mathbf{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i)^{-1}\mathbf{X}_i'\mathbf{\Sigma}_i^{-1}\mathbf{Z}_i] \\
&\quad \times (\mathbf{Z}_i'\mathbf{\Sigma}_i^{-1}\mathbf{Z}_i)^{-1}, \\
\mathbf{Q}_{31} &= -(\mathbf{Z}_i'\mathbf{\Sigma}_i^{-1}\mathbf{Z}_i)^{-1}[\mathbf{Z}_i'\mathbf{\Sigma}_i^{-1}\mathbf{X}_i - \mathbf{Z}_i'\mathbf{\Sigma}_i^{-1}\mathbf{W}_i\mathbf{Q}_{22}\mathbf{W}_i'\mathbf{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i + \\
&\quad \mathbf{Z}_i'\mathbf{\Sigma}_i^{-1}\mathbf{X}_i(\mathbf{X}_i'\mathbf{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i)^{-1}\mathbf{X}_i'\mathbf{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i\mathbf{Q}_{22}\mathbf{W}_i'\mathbf{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i] \\
&\quad \times (\mathbf{X}_i'\mathbf{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i)^{-1}, \\
\mathbf{Q}_{32} &= -(\mathbf{Z}_i'\mathbf{\Sigma}_i^{-1}\mathbf{Z}_i)^{-1}[\mathbf{Z}_i'\mathbf{\Sigma}_i^{-1}\mathbf{W}_i - \mathbf{Z}_i'\mathbf{\Sigma}_i^{-1}\mathbf{X}_i(\mathbf{X}_i'\mathbf{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i)^{-1} \\
&\quad \times \mathbf{X}_i'\mathbf{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i]\mathbf{Q}_{22}, \\
\mathbf{Q}_{33} &= (\mathbf{Z}_i'\mathbf{\Sigma}_i^{-1}\mathbf{Z}_i)^{-1} + (\mathbf{Z}_i'\mathbf{\Sigma}_i^{-1}\mathbf{Z}_i)^{-1}[\mathbf{Z}_i'\mathbf{\Sigma}_i^{-1}\mathbf{W}_i\mathbf{Q}_{22}\mathbf{W}_i'\mathbf{\Sigma}_i^{-1}\mathbf{Z}_i \\
&\quad - \mathbf{Z}_i'\mathbf{\Sigma}_i^{-1}\mathbf{W}_i\mathbf{Q}_{22}\mathbf{W}_i'\mathbf{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i(\mathbf{X}_i'\mathbf{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i)^{-1}\mathbf{X}_i'\mathbf{\Sigma}_i^{-1}\mathbf{Z}_i \\
&\quad - \mathbf{Z}_i'\mathbf{\Sigma}_i^{-1}\mathbf{X}_i(\mathbf{X}_i'\mathbf{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i)^{-1}\mathbf{X}_i'\mathbf{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i\mathbf{Q}_{22}\mathbf{W}_i'\mathbf{\Sigma}_i^{-1}\mathbf{Z}_i \\
&\quad + \mathbf{Z}_i'\mathbf{\Sigma}_i^{-1}\mathbf{X}_i(\mathbf{X}_i'\mathbf{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i)^{-1}\mathbf{X}_i'\mathbf{\Sigma}_i^{-1}\mathbf{Z}_i \\
&\quad + \mathbf{Z}_i'\mathbf{\Sigma}_i^{-1}\mathbf{X}_i(\mathbf{X}_i'\mathbf{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i)^{-1}\mathbf{X}_i'\mathbf{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{W}_i\mathbf{Q}_{22}\mathbf{W}_i'\mathbf{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i \\
&\quad \times (\mathbf{X}_i'\mathbf{\Sigma}_i^{-1}\mathbf{M}_{Z_i}^{\Sigma_i^{-1}}\mathbf{X}_i)^{-1}\mathbf{X}_i'\mathbf{\Sigma}_i^{-1}\mathbf{Z}_i](\mathbf{Z}_i'\mathbf{\Sigma}_i^{-1}\mathbf{Z}_i)^{-1},
\end{aligned}
$$

respectively. Regarding that

$$
\begin{aligned}
\widehat{\boldsymbol{\beta}}^{(i)} &= \mathbf{Q}_{11}\mathbf{X}_i'\mathbf{\Sigma}_i^{-1}\mathbf{Y}_i + \mathbf{Q}_{12}\mathbf{W}_i'\mathbf{\Sigma}_i^{-1}\mathbf{Y}_i + \mathbf{Q}_{13}\mathbf{Z}_i'\mathbf{\Sigma}_i^{-1}\mathbf{Y}_i, \\
\widehat{\gamma_i}^{(i)} &= \mathbf{Q}_{21}\mathbf{X}_i'\mathbf{\Sigma}_i^{-1}\mathbf{Y}_i + \mathbf{Q}_{22}\mathbf{W}_i'\mathbf{\Sigma}_i^{-1}\mathbf{Y}_i + \mathbf{Q}_{23}\mathbf{Z}_i'\mathbf{\Sigma}_i^{-1}\mathbf{Y}_i, \\
\widehat{\boldsymbol{\kappa}_i}^{(i)} &= \mathbf{Q}_{31}\mathbf{X}_i'\mathbf{\Sigma}_i^{-1}\mathbf{Y}_i + \mathbf{Q}_{32}\mathbf{W}_i'\mathbf{\Sigma}_i^{-1}\mathbf{Y}_i + \mathbf{Q}_{33}\mathbf{Z}_i'\mathbf{\Sigma}_i^{-1}\mathbf{Y}_i,
\end{aligned}
$$

$i = 1, 2$, the proof is complete. □

Notation 1 The model (3) can be rewritten as

$$
\mathbf{Y} = (\mathbf{W}, \mathbf{Z})\begin{pmatrix} \boldsymbol{\delta} \\ \boldsymbol{\kappa} \end{pmatrix} + \boldsymbol{\varepsilon}, \tag{6}
$$

where

$$
\mathbf{Y} = \begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix},\ \mathbf{W} = \begin{pmatrix} \mathbf{X}_1 & \mathbf{W}_1 & \mathbf{0} \\ \mathbf{X}_2 & \mathbf{0} & \mathbf{W}_2 \end{pmatrix},\ \mathbf{Z} = \begin{pmatrix} \mathbf{Z}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{Z}_2 \end{pmatrix},\ \boldsymbol{\delta} = \begin{pmatrix} \boldsymbol{\beta} \\ \gamma \end{pmatrix},\ \boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}
$$

and

$$\mathbf{\Sigma} = \mathrm{var}(\mathbf{Y}) = \begin{pmatrix} \mathbf{\Sigma}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{\Sigma}_2 \end{pmatrix},$$

so we get the (ordinary) linear model with nuisance parameters.

Proposition 1 *In the regular model (6) the BLUE of the parameter $(\boldsymbol{\delta}', \boldsymbol{\kappa}')'$ is given as*

$$\begin{pmatrix} \widehat{\boldsymbol{\delta}} \\ \widehat{\boldsymbol{\kappa}} \end{pmatrix} = \begin{pmatrix} (\mathbf{W}'\mathbf{\Sigma}^{-1}\mathbf{M}_Z^{\Sigma^{-1}}\mathbf{W})^{-1}\mathbf{W}'\mathbf{\Sigma}^{-1}\mathbf{M}_Z^{\Sigma^{-1}} \\ (\mathbf{Z}'\mathbf{\Sigma}^{-1}\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{\Sigma}^{-1}\mathbf{M}_W^{\Sigma^{-1}M_Z^{\Sigma^{-1}}} \end{pmatrix} \mathbf{Y}. \tag{7}$$

Proof See [3, Theorem 1].

Theorem 4 *In the regular model (3) the BLUEs of the parameters $\boldsymbol{\beta}, \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2$ are given as*

$$\begin{aligned}
\widehat{\boldsymbol{\beta}} &= (\mathbf{X}_1'\mathbf{\Sigma}_1^{-1}\mathbf{M}_{Z_1}^{\Sigma_1^{-1}}\mathbf{M}_{W_1}^{\Sigma_1^{-1}M_{Z_1}^{\Sigma_1^{-1}}}\mathbf{X}_1 + \mathbf{X}_2'\mathbf{\Sigma}_2^{-1}\mathbf{M}_{Z_2}^{\Sigma_2^{-1}}\mathbf{M}_{W_2}^{\Sigma_2^{-1}M_{Z_2}^{\Sigma_2^{-1}}}\mathbf{X}_2)^{-1} \\
&\quad \times (\mathbf{X}_1'\mathbf{\Sigma}_1^{-1}\mathbf{M}_{Z_1}^{\Sigma_1^{-1}}\mathbf{M}_{W_1}^{\Sigma_1^{-1}M_{Z_1}^{\Sigma_1^{-1}}}\mathbf{Y}_1 + \mathbf{X}_2'\mathbf{\Sigma}_2^{-1}\mathbf{M}_{Z_2}^{\Sigma_2^{-1}}\mathbf{M}_{W_2}^{\Sigma_2^{-1}M_{Z_2}^{\Sigma_2^{-1}}}\mathbf{Y}_2), \\
\widehat{\boldsymbol{\gamma}_1} &= (\mathbf{W}_1'\mathbf{\Sigma}_1^{-1}\mathbf{M}_{Z_1}^{\Sigma_1^{-1}}\mathbf{W}_1)^{-1}\mathbf{W}_1'\mathbf{\Sigma}_1^{-1}\mathbf{M}_{Z_1}^{\Sigma_1^{-1}}(\mathbf{Y}_1 - \mathbf{X}_1\widehat{\boldsymbol{\beta}}), \\
\widehat{\boldsymbol{\gamma}_2} &= (\mathbf{W}_2'\mathbf{\Sigma}_2^{-1}\mathbf{M}_{Z_2}^{\Sigma_2^{-1}}\mathbf{W}_2)^{-1}\mathbf{W}_2'\mathbf{\Sigma}_2^{-1}\mathbf{M}_{Z_2}^{\Sigma_2^{-1}}(\mathbf{Y}_2 - \mathbf{X}_2\widehat{\boldsymbol{\beta}}), \\
\widehat{\boldsymbol{\kappa}_1} &= (\mathbf{Z}_1'\mathbf{\Sigma}_1^{-1}\mathbf{Z}_1)^{-1}\mathbf{Z}_1'\mathbf{\Sigma}_1^{-1}(\mathbf{Y}_1 - \mathbf{X}_1\widehat{\boldsymbol{\beta}} - \mathbf{W}_1\widehat{\boldsymbol{\gamma}_1}), \\
\widehat{\boldsymbol{\kappa}_2} &= (\mathbf{Z}_2'\mathbf{\Sigma}_2^{-1}\mathbf{Z}_2)^{-1}\mathbf{Z}_2'\mathbf{\Sigma}_2^{-1}(\mathbf{Y}_2 - \mathbf{X}_2\widehat{\boldsymbol{\beta}} - \mathbf{W}_2\widehat{\boldsymbol{\gamma}_2}).
\end{aligned}$$

Proof According to Notation 1 we can use (7) to get the result. Here

$$\mathbf{\Sigma}^{-1}\mathbf{M}_Z^{\Sigma^{-1}} = \mathbf{\Sigma}^{-1} - \mathbf{\Sigma}^{-1}\mathbf{Z}(\mathbf{Z}'\mathbf{\Sigma}^{-1}\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{\Sigma}^{-1} = \begin{pmatrix} \mathbf{\Sigma}_1^{-1}\mathbf{M}_{Z_1}^{\Sigma_1^{-1}} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Sigma}_2^{-1}\mathbf{M}_{Z_2}^{\Sigma_2^{-1}} \end{pmatrix}$$

thus (we have used Corollary 1)

$$(\mathbf{W}'\mathbf{\Sigma}^{-1}\mathbf{M}_Z^{\Sigma^{-1}}\mathbf{W})^{-1} =$$

$$= \begin{pmatrix} \mathbf{X}_1'\mathbf{\Sigma}_1^{-1}\mathbf{M}_{Z_1}^{\Sigma_1^{-1}}\mathbf{X}_1 + \mathbf{X}_2'\mathbf{\Sigma}_2^{-1}\mathbf{M}_{Z_2}^{\Sigma_2^{-1}}\mathbf{X}_2 & \mathbf{X}_1'\mathbf{\Sigma}_1^{-1}\mathbf{M}_{Z_1}^{\Sigma_1^{-1}}\mathbf{W}_1 & \mathbf{X}_2'\mathbf{\Sigma}_2^{-1}\mathbf{M}_{Z_2}^{\Sigma_2^{-1}}\mathbf{W}_2 \\ \mathbf{W}_1'\mathbf{\Sigma}_1^{-1}\mathbf{M}_{Z_1}^{\Sigma_1^{-1}}\mathbf{X}_1 & \mathbf{W}_1'\mathbf{\Sigma}_1^{-1}\mathbf{M}_{Z_1}^{\Sigma_1^{-1}}\mathbf{W}_1 & \mathbf{0} \\ \mathbf{W}_2'\mathbf{\Sigma}_2^{-1}\mathbf{M}_{Z_2}^{\Sigma_2^{-1}}\mathbf{X}_2 & \mathbf{0} & \mathbf{W}_2'\mathbf{\Sigma}_2^{-1}\mathbf{M}_{Z_2}^{\Sigma_2^{-1}}\mathbf{W}_2 \end{pmatrix}^{-1}$$

$$\overset{\text{C.1}}{=} \begin{pmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} & \mathbf{Q}_{13} \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} & \mathbf{Q}_{23} \\ \mathbf{Q}_{31} & \mathbf{Q}_{32} & \mathbf{Q}_{33} \end{pmatrix},$$

where

$$\begin{aligned}
\mathbf{Q}_{11} &= (\mathbf{X}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{M}_{Z_1}^{\Sigma_1^{-1}}\mathbf{M}_{W_1}^{\Sigma_1^{-1}M_{Z_1}^{\Sigma_1^{-1}}}\mathbf{X}_1 + \mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{M}_{Z_2}^{\Sigma_2^{-1}}\mathbf{M}_{W_2}^{\Sigma_2^{-1}M_{Z_2}^{\Sigma_2^{-1}}}\mathbf{X}_2)^{-1},\\
\mathbf{Q}_{12} &= -\mathbf{Q}_{11}\mathbf{X}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{M}_{Z_1}^{\Sigma_1^{-1}}\mathbf{W}_1(\mathbf{W}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{M}_{Z_1}^{\Sigma_1^{-1}}\mathbf{W}_1)^{-1},\\
\mathbf{Q}_{13} &= -\mathbf{Q}_{11}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{M}_{Z_2}^{\Sigma_2^{-1}}\mathbf{W}_2(\mathbf{W}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{M}_{Z_2}^{\Sigma_2^{-1}}\mathbf{W}_2)^{-1},\\
\mathbf{Q}_{21} &= -(\mathbf{W}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{M}_{Z_1}^{\Sigma_1^{-1}}\mathbf{W}_1)^{-1}\mathbf{W}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{M}_{Z_1}^{\Sigma_1^{-1}}\mathbf{X}_1\mathbf{Q}_{11},\\
\mathbf{Q}_{22} &= (\mathbf{W}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{M}_{Z_1}^{\Sigma_1^{-1}}\mathbf{W}_1)^{-1} + (\mathbf{W}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{M}_{Z_1}^{\Sigma_1^{-1}}\mathbf{W}_1)^{-1}\mathbf{W}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{M}_{Z_1}^{\Sigma_1^{-1}}\mathbf{X}_1\\
&\quad \times \mathbf{Q}_{11}\mathbf{X}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{M}_{Z_1}^{\Sigma_1^{-1}}\mathbf{W}_1(\mathbf{W}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{M}_{Z_1}^{\Sigma_1^{-1}}\mathbf{W}_1)^{-1},\\
\mathbf{Q}_{23} &= (\mathbf{W}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{M}_{Z_1}^{\Sigma_1^{-1}}\mathbf{W}_1)^{-1}\mathbf{W}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{M}_{Z_1}^{\Sigma_1^{-1}}\mathbf{X}_1\\
&\quad \times \mathbf{Q}_{11}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{M}_{Z_2}^{\Sigma_2^{-1}}\mathbf{W}_2(\mathbf{W}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{M}_{Z_2}^{\Sigma_2^{-1}}\mathbf{W}_2)^{-1},\\
\mathbf{Q}_{31} &= -(\mathbf{W}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{M}_{Z_2}^{\Sigma_2^{-1}}\mathbf{W}_2)^{-1}\mathbf{W}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{M}_{Z_2}^{\Sigma_2^{-1}}\mathbf{X}_2\mathbf{Q}_{11},\\
\mathbf{Q}_{32} &= (\mathbf{W}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{M}_{Z_2}^{\Sigma_2^{-1}}\mathbf{W}_2)^{-1}\mathbf{W}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{M}_{Z_2}^{\Sigma_2^{-1}}\mathbf{X}_2\\
&\quad \times \mathbf{Q}_{11}\mathbf{X}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{M}_{Z_1}^{\Sigma_1^{-1}}\mathbf{W}_1(\mathbf{W}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{M}_{Z_1}^{\Sigma_1^{-1}}\mathbf{W}_1)^{-1},\\
\mathbf{Q}_{33} &= (\mathbf{W}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{M}_{Z_2}^{\Sigma_2^{-1}}\mathbf{W}_2)^{-1} + (\mathbf{W}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{M}_{Z_2}^{\Sigma_2^{-1}}\mathbf{W}_2)^{-1}\mathbf{W}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{M}_{Z_2}^{\Sigma_2^{-1}}\mathbf{X}_2\\
&\quad \times \mathbf{Q}_{11}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{M}_{Z_2}^{\Sigma_2^{-1}}\mathbf{W}_2(\mathbf{W}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{M}_{Z_2}^{\Sigma_2^{-1}}\mathbf{W}_2)^{-1}.
\end{aligned}$$

Utilizing that

$$\mathbf{W}'\boldsymbol{\Sigma}^{-1}\mathbf{M}_Z^{\Sigma^{-1}} = \begin{pmatrix} \mathbf{X}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{M}_{Z_1}^{\Sigma_1^{-1}} & \mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{M}_{Z_2}^{\Sigma_2^{-1}} \\ \mathbf{W}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{M}_{Z_1}^{\Sigma_1^{-1}} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{M}_{Z_2}^{\Sigma_2^{-1}} \end{pmatrix},$$

we get (after some calculations) the BLUEs of the useful parameters $\boldsymbol{\beta}, \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2$. To get the same for the nuisance parameters $\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2$ it is sufficient to realize that

$$(\mathbf{Z}'\boldsymbol{\Sigma}^{-1}\mathbf{Z})^{-1}\mathbf{Z}'\boldsymbol{\Sigma}^{-1} = \begin{pmatrix} (\mathbf{Z}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{Z}_1)^{-1}\mathbf{Z}_1'\boldsymbol{\Sigma}_1^{-1} & \mathbf{0} \\ \mathbf{0} & (\mathbf{Z}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{Z}_2)^{-1}\mathbf{Z}_2'\boldsymbol{\Sigma}_2^{-1} \end{pmatrix}$$

and

$$\begin{aligned}
\mathbf{M}_W^{\Sigma^{-1}M_Z^{\Sigma^{-1}}}\mathbf{Y} &= \mathbf{Y} - \mathbf{W}(\mathbf{W}'\boldsymbol{\Sigma}^{-1}\mathbf{M}_Z^{\Sigma^{-1}}\mathbf{W})^{-1}\mathbf{W}'\boldsymbol{\Sigma}^{-1}\mathbf{M}_Z^{\Sigma^{-1}}\mathbf{Y}\\
&= \begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix} - \begin{pmatrix} \mathbf{X}_1 & \mathbf{W}_1 & \mathbf{0} \\ \mathbf{X}_2 & \mathbf{0} & \mathbf{W}_2 \end{pmatrix} \begin{pmatrix} \widehat{\boldsymbol{\beta}} \\ \widehat{\boldsymbol{\gamma}_1} \\ \widehat{\boldsymbol{\gamma}_2} \end{pmatrix} = \begin{pmatrix} \mathbf{Y}_1 - \mathbf{X}_1\widehat{\boldsymbol{\beta}} - \mathbf{W}_1\widehat{\boldsymbol{\gamma}_1} \\ \mathbf{Y}_2 - \mathbf{X}_2\widehat{\boldsymbol{\beta}} - \mathbf{W}_2\widehat{\boldsymbol{\gamma}_2} \end{pmatrix}. \qquad \square
\end{aligned}$$

Remark 4 Regarding that $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$ are supposed to be positive definite, we can write (see [2, Lemma 10.1.35, p. 441])

$$\begin{aligned}\mathbf{\Sigma}_1^{-1}\mathbf{M}_{Z_1}^{\Sigma_1^{-1}} &= \mathbf{\Sigma}_1^{-1} - \mathbf{\Sigma}_1^{-1}\mathbf{Z}_1(\mathbf{Z}_1'\mathbf{\Sigma}_1^{-1}\mathbf{Z}_1)^{-1}\mathbf{Z}_1'\mathbf{\Sigma}_1^{-1} = (\mathbf{M}_{Z_1}\mathbf{\Sigma}_1\mathbf{M}_{Z_1})^+,\\ \mathbf{\Sigma}_2^{-1}\mathbf{M}_{Z_2}^{\Sigma_2^{-1}} &= \mathbf{\Sigma}_2^{-1} - \mathbf{\Sigma}_2^{-1}\mathbf{Z}_2(\mathbf{Z}_2'\mathbf{\Sigma}_2^{-1}\mathbf{Z}_2)^{-1}\mathbf{Z}_2'\mathbf{\Sigma}_2^{-1} = (\mathbf{M}_{Z_2}\mathbf{\Sigma}_2\mathbf{M}_{Z_2})^+,\end{aligned}$$

respectively.

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Some Stability Theorems for Some Iteration Processes

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Abstract

In this paper, we obtain some stability results for Picard and Mann iteration processes in metric space and normed linear space respectively, using two different contractive definitions which are more general than those of Harder and Hicks [4], Rhoades [10, 11], Osilike [8], Osilike and Udomene [9], Berinde [1, 2], Imoru and Olatinwo [5] and Imoru et al [6].

Our results are generalizations of some results of Harder and Hicks [4], Rhoades [10, 11], Osilike [8], Osilike and Udomene [9], Berinde [1, 2], Imoru and Olatinwo [5] and Imoru et al [6].

Key words: Stability results; Picard and Mann iteration processes.

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1 Introduction

Let (X, d) be a complete metric space, $T : X \to X$ a selfmap of X. Suppose that $F_T = \{p \in X \mid Tp = p\}$ is the set of fixed points of T. Let $\{x_n\}_{n=0}^{\infty} \subset X$ be the sequence generated by an iteration procedure involving T which is defined by

$$x_{n+1} = f(T, x_n), \quad n = 0, 1, 2, \dots \tag{1}$$

where $x_0 \in X$ is the initial approximation and f is some function. Suppose $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point p of T. Let $\{y_n\}_{n=0}^{\infty} \subset X$ and set

$$\epsilon_n = d(y_{n+1}, f(T, y_n)), \quad n = 0, 1, 2, \dots$$

Then, the iteration procedure (1) is said to be T-stable or stable with respect to T if and only if $\lim_{n\to\infty} \epsilon_n = 0$ implies $\lim_{n\to\infty} y_n = p$. If in (1),

$$f(T, x_n) = Tx_n, \quad n = 0, 1, 2, \ldots,$$

then we have the Picard iteration process, while we obtain the Mann iteration if

$$f(T, x_n) = (1 - \alpha_n)x_n + \alpha_n Tx_n, \quad n = 0, 1, 2, \ldots, \ \alpha_n \in [0, 1].$$

Several stability results have been obtained by various authors using different contractive definitions. Harder and Hicks [4] obtained interesting stability results for some iteration procedures using various contractive definitions. Rhoades [10,11] generalized the results of Harder and Hicks [4] to a more general contractive mapping. In Osilike [8], a generalization of some of the results of Harder and Hicks [4] and Rhoades [11] was obtained by employing the following contractive definition: there exist a constant $L \geq 0$ and $a \in [0, 1)$ such $\forall x, y \in X$,

$$d(Tx, Ty) \leq Ld(x, Tx) + ad(x, y). \tag{2}$$

Condition(2) is more general than those of Rhoades[11] and Harder and Hicks[4]. As in Harder and Hicks [4], Berinde [1] obtained the same stability results for the same iteration procedures using the same contractive definitions, but applied a different method. The method of Berinde [1] is similar to that employed in Osilike and Udomene [9] .

Recently, Imoru and Olatinwo [5] obtained some stability results for Picard and Mann iteration procedures by using a more general contractive condition than those of Harder and Hicks [4], Rhoades [11], Osilike [8], Osilike and Udomene [9] and Berinde [1]. In the paper [5], the following contractive definition was employed: there exist $a \in [0, 1)$ and a monotone increasing function $\phi : \Re_+ \to \Re_+$, with $\phi(0) = 0$, such that $\forall x, y \in X$,

$$d(Tx, Ty) \leq \phi(d(x, Tx)) + ad(x, y). \tag{3}$$

A function $h : \Re_+ \to \Re_+$ is called a comparison function if:
(i) h is monotone increasing;
(ii) $\lim_{n\to\infty} h^n(t) = 0, \ \forall t \geq 0$.
We remark here that every comparison function satisfies the condition $h(0) = 0$.

It is our purpose in this paper to obtain some stability results by applying two different contractive definitions using again the method of Berinde [1]. We shall use the following contractive definitions:
I) there exist a constant $a \in [0, 1)$ and a monotone increasing function $\Phi : \Re_+ \to \Re_+$ with $\Phi(0) = 1$, such that $\forall x, y \in X$,

$$d(Tx, Ty) \leq ad(x, y)\Phi(d(x, Tx)), \tag{4}$$

II) there exist a constant $L \geq 0$ and a function $\Psi : \Re_+ \to \Re_+$ such that $\forall x, y \in X$,

$$d(Tx, Ty) \leq \Psi(d(x, y))e^{Ld(x,Tx)} \tag{5}$$

where Ψ may be a comparison function or just a monotone increasing function. The contractive conditions (4) and (5) are independent as the right-hand side of (4) cannot be obtained from the right-hand side of (5) or vice-versa.

Condition (4) is more general than (2) in the following sense: If in (4),

$$\Phi(u) = 1 + \frac{ku}{d(x,y)}, \quad k \geq 0,\ d(x,y) \neq 0,\ \forall x, y \in X,\ x \neq y,\ u \in \Re_+,$$

then we obtain the condition (2).

Also, if in (4), we have

$$\Phi(u) = 1 + \frac{\phi(u)}{d(x,y)}, \quad d(x,y) \neq 0,\ \forall x, y \in X,\ x \neq y,\ u \in \Re_+,$$

where ϕ is also a monotone increasing function, then we obtain condition (3). Also, if $\Phi(u) = 1$, $\forall u \in \Re_+$, then we have the strict contraction employed in Harder and Hicks [4], Zeidler [13] and Berinde [1,2].

Similarly, condition (5) is more general than (2) in the sense that if in (5),

$$\Psi(u) = (au + Ld(x,Tx))e^{-Ld(x,Tx)}, \quad a \in [0,1),\ L \geq 0,\ u \in \Re_+,\ \forall x \in X,$$

and if Ψ is monotone increasing, then we obtain the condition(2).

Again, if $\Psi(u) = au$, $a \in [0,1)$, $u \in \Re_+$ and $L = 0$ in (5), then we get the strict contraction employed in Harder and Hicks [4], Berinde [1,2] and also in the classical Banach's contraction mapping principle discussed in Zeidler [13] and other standard texts on the fixed point theory.

Moreover , if in (5),

$$\Psi(u) = (\psi(u) + Ld(x,Tx))e^{-Ld(x,Tx)}, \quad \forall x \in X,\ u \in \Re_+,\ L \geq 0,$$

and if Ψ is monotone increasing and ψ is a comparison function, then we obtain the contractive mapping of Imoru et al [6].

However, we obtain the contractive definition employed in the extension of the Banach's contraction mapping principle due to Berinde [3] if $L = 0$ in (5). See also Berinde [2] for detail on the various generalizations of the Picard–Banach–Caccioppoli theorem.

We shall employ the following Lemmas in the sequel.

Lemma 1 (Berinde [1]) *If δ is a real number such that $0 \leq \delta < 1$, and $\{\epsilon_n\}_{n=0}^{\infty}$ is a sequence of positive numbers such that $\lim_{n\to\infty} \epsilon_n = 0$, then for any sequence of positive numbers $\{u_n\}_{n=0}^{\infty}$ satisfying*

$$u_{n+1} \leq \delta u_n + \epsilon_n, \quad n = 0, 1, \ldots \tag{6}$$

we have

$$\lim_{n\to\infty} u_n = 0.$$

Remark 1 The proof of this lemma is contained in Berinde [1].

Lemma 2 *If $\psi : \Re_+ \to \Re_+$ is a subadditive comparison function and $\{\epsilon_n\}_{n=0}^{\infty}$ is a sequence of positive numbers such that $\lim_{n\to\infty} \epsilon_n = 0$, then for any sequence of positive numbers $\{u_n\}_{n=0}^{\infty}$ satisfying*

$$u_{n+1} \leq \sum_{m=0}^{s} \delta_m \psi^m(u_n) + \epsilon_n, \quad n = 0, 1, 2, \ldots, \tag{7}$$

where $\sum_{m=0}^{s} \delta_m = 1$, $\delta_0, \delta_1, \ldots, \delta_s \in [0, 1]$, we have

$$\lim_{n\to\infty} u_n = 0.$$

Remark 2 The proof of this Lemma is contained in Imoru et al [6].

Remark 3 If $\delta_k = 0$ in (7), $k = 1, 2, \ldots, s$, then we obtain the Lemma 1 of Berinde [1] with $0 \leq \delta_o < 1$.

Remark 4 If $\delta_1 = 1$ and $\delta_o = \delta_2 = \delta_3 = \cdots = \delta_{s-1} = \delta_s = 0$ in (7), then we obtain a stability result for the Picard iteration process.

Remark 5 We have a stability result for the Krasnoselskij iteration procedure if $\delta_o = \delta_1 = 1/2$ and $\delta_2 = \delta_3 = \cdots = \delta_s = 0$ in (7).

Remark 6 We obtain stability results for the Mann and Schaefer's iteration processes if $\delta_0 + \delta_1 = 1$, $\delta_2 = \delta_3 = \cdots = \delta_s = 0$ in (7).

Remark 7 If $\delta_0 + \delta_1 + \delta_2 = 1$, $\delta_3 = \delta_4 = \cdots = \delta_s = 0$ in (7), then we obtain a stability result for the Ishikawa iteration procedure.

Remark 8 If $\sum_{m=0}^{k} \delta_m = 1$ (i.e. $s = k$) in (7),then we have a stability result for the Kirk's iteration process.

2 Main Results

The following are stability results for the Picard iteration process.

Theorem 1 *Let (X, d) be a complete metric space and $T : X \to X$ a selfmap of X satisfying (4). Suppose T has a fixed point p. Let $x_0 \in X$ and let*

$$x_{n+1} = f(T, x_n) = Tx_n, \quad n = 0, 1, \ldots$$

be the Picard iteration associated to T. Suppose also that $\Phi : \Re_+ \to \Re_+$ is a monotone increasing function such that $\Phi(0) = 1$. Then, the Picard iteration is T-stable.

Proof Let $\epsilon_n = d(y_{n+1}, Ty_n)$, $n = 0, 1, \ldots$ and suppose $\lim_{n\to\infty} \epsilon_n = 0$. Then, we shall establish that $\lim_{n\to\infty} y_n = p$. using (4) and the triangle inequality. Therefore,

$$\begin{aligned} d(y_{n+1}, p) &\le d(y_{n+1}, Ty_n) + d(Ty_n, p) \\ = \epsilon_n + d(Ty_n, Tp) &= d(Tp, Ty_n) + \epsilon_n \le ad(p, y_n)\Phi(d(p, Tp)) + \epsilon_n \\ &= ad(y_n, p)\Phi(0) + \epsilon_n = ad(y_n, p) + \epsilon_n. \end{aligned} \tag{8}$$

Since $a \in [0, 1)$, using Lemma 1 in (8) yields $\lim_{n\to\infty} y_n = p$.

Conversely, let $\lim_{n\to\infty} y_n = p$. Then, by (4) and the triangle inequality, we have

$$\begin{aligned} \epsilon_n = d(y_{n+1}, Ty_n) &\le d(y_{n+1}, p) + d(p, Ty_n) = d(y_{n+1}, p) + d(Tp, Ty_n) \\ &\le d(y_{n+1}, p) + ad(p, y_n)\Phi(d(p, Tp)) = d(y_{n+1}, p) + ad(y_n, p)\Phi(0) \\ &= d(y_{n+1}, p) + ad(y_n, p) \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

□

Theorem 2 *Let (X, d) be a complete metric space and $T : X \to X$ a selfmap of X satisfying (5). Suppose that T has a fixed point p. Let $x_0 \in X$ and let*

$$x_{n+1} = f(T, x_n) = Tx_n, \quad n = 0, 1, \ldots,$$

be the Picard iteration associated to T. Suppose that $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function (or just a monotone increasing function) which is continuous. Then, the Picard iteration is T-stable.

Proof Let $\epsilon_n = d(y_{n+1}, Ty_n)$, $n = 0, 1, \ldots$, and suppose that $\lim_{n\to\infty} \epsilon_n = 0$. Then, we shall establish that $\lim_{n\to\infty} y_n = p$, using (5) and the triangle inequality. Therefore,

$$\begin{aligned} d(y_{n+1}, p) &\le d(y_{n+1}, Ty_n) + d(Ty_n, p) \\ = \epsilon_n + d(Ty_n, Tp) &= d(Tp, Ty_n) + \epsilon_n \le \Psi(d(p, y_n))e^{Ld(p,Tp)} + \epsilon_n \\ &= \Psi(d(y_n, p)) + \epsilon_n. \end{aligned} \tag{9}$$

Applying Lemma 2 in (9) yields $\lim_{n\to\infty} y_n = p$.

Conversely, let $\lim_{n\to\infty} y_n = p$. Then, by (5)and the triangle inequality, we obtain

$$\begin{aligned} \epsilon_n = d(y_{n+1}, Ty_n) &\le d(y_{n+1}, p) + d(p, Ty_n) = d(y_{n+1}, p) + d(Tp, Ty_n) \\ &\le d(y_{n+1}, p) + \Psi(d(p, y_n))e^{Ld(p,Tp)} \\ &= d(y_{n+1}, p) + \Psi(d(y_n, p)) \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

□

Remark 9 Theorem 1 is a generalization of Theorem 3.1 of Imoru and Olatinwo [5], while Theorem 2 is a generalization of both Theorems P1 and P2 of Imoru et al [6]. Also, each of the Theorem 3.1 of [5] and Theorems P1 and P2 of [6] is itself a generalization of Theorem 2 of Harder and Hicks [4], Theorem 1 of Rhoades[10, 11], Theorems 1 and 2 of Berinde [1], Theorem 1 of Osilike [8] as well as Theorem 4 of Osilike and Udomene [9].

We now establish some stability results for the Mann iteration process.

Theorem 3 *Let* $(X, ||\cdot||)$ *be a normed linear space, and* $T : X \to X$ *a selfmap of* X *satisfying (4). Suppose* T *has a fixed point* p. *Let* $x_o \in X$ *and let*

$$x_{n+1} = f(T, x_n) = (1-\alpha_n)x_n + \alpha_n Tx_n, \quad \alpha_n \in [0,1],\ n = 0,1,\ldots,$$

be the Mann iteration process such that $0 < \alpha \leq \alpha_n$, $n = 0,1,2,\ldots$ *Let* $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ *be a monotone increasing function such that* $\Phi(0) = 1$. *Then, the Mann iteration process is T-stable.*

Proof Let $\epsilon_n = ||y_{n+1} - (1-\alpha_n)y_n - \alpha_n Ty_n||$, $n = 0,1,\ldots$ and suppose that $\lim_{n\to\infty} \epsilon_n = 0$. Then,we shall prove that $\lim_{n\to\infty} y_n = p$, by (4) and the triangle inequality: Therefore,

$$\begin{aligned}
||y_{n+1} - p|| &\leq ||y_{n+1} - (1-\alpha_n)y_n - \alpha_n Ty_n|| + ||(1-\alpha_n)y_n + \alpha_n Ty_n - p|| \\
&= \epsilon_n + ||(1-\alpha_n)y_n + \alpha_n Ty_n - (1-\alpha_n+\alpha_n)p|| \\
&= ||(1-\alpha_n)(y_n - p) + \alpha_n(Ty_n - p)|| + \epsilon_n \\
&\leq (1-\alpha_n)||y_n - p|| + \alpha_n||Ty_n - p|| + \epsilon_n \\
&= (1-\alpha_n)||y_n - p|| + \alpha_n||Ty_n - Tp|| + \epsilon_n \\
&\leq (1-\alpha_n)||y_n - p|| + \alpha_n a||p - y_n||\Phi(||p - Tp||) + \epsilon_n \\
&= (1-\alpha_n)||y_n - p|| + \alpha_n\ a||y_n - p||\Phi(0) + \epsilon_n \\
&= (1-\alpha_n)||y_n - p|| + a\alpha_n||y_n - p|| + \epsilon_n \\
&= [1-(1-a)\alpha_n]||y_n - p|| + \epsilon_n \\
&\leq [1-(1-a)\alpha]||y_n - p|| + \epsilon_n. \qquad (10)
\end{aligned}$$

Using Lemma 1 in (10) since $0 \leq 1-(1-a)\alpha < 1$, we obtain

$$\lim_{n\to\infty} y_n = p.$$

Conversely, let $\lim_{n\to\infty} y_n = p$. Then, by (4) and the triangle inequality, we get

$$\begin{aligned}
\epsilon_n &= ||y_{n+1} - (1-\alpha_n)y_n - \alpha_n Ty_n|| \\
&\leq ||y_{n+1} - p|| + ||p - (1-\alpha_n)y_n - \alpha_n Ty_n|| \\
&= ||y_{n+1} - p|| + ||(1-\alpha_n+\alpha_n)p - (1-\alpha_n)y_n - \alpha_n Ty_n|| \\
&= ||y_{n+1} - p|| + ||(1-\alpha_n)(p - y_n) + \alpha_n(p - Ty_n)|| \\
&\leq ||y_{n+1} - p|| + (1-\alpha_n)||y_n - p|| + \alpha_n||p - Ty_n|| \\
&= ||y_{n+1} - p|| + (1-\alpha_n)||y_n - p|| + \alpha_n||Tp - Ty_n|| \\
&\leq ||y_{n+1} - p|| + (1-\alpha_n)||y_n - p|| + \alpha_n a||p - y_n||\Phi(||p - Tp||) \\
&= ||y_{n+1} - p|| + (1-\alpha_n)||y_n - p|| + \alpha_n a||y_n - p||\phi(0) \\
&= ||y_{n+1} - p|| + [1-(1-a)\alpha_n]||y_n - p|| \\
&\leq ||y_{n+1} - p|| + [1-(1-a)\alpha]||y_n - p|| \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

□

Theorem 4 *Let $(X, ||\cdot||)$ be a normed linear space and $T : X \to X$ a selfmap of X satisfying (5). Suppose that T has a fixed point p. Let $x_0 \in X$ and let $x_{n+1} = f(T, x_n) = (1-\alpha_n)x_n + \alpha_n Tx_n$, $\alpha_n \in [0,1]$, $n = 0,1,\ldots,$ be the Mann iteration process such that $0 < \alpha \leq \alpha_n$, $n = 0,1,2,\ldots$ Suppose that $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function (or just a monotone increasing function) which is continuous. Then, the Mann iteration is T-stable.*

Proof Let $\epsilon_n = ||y_{n+1} - (1-\alpha_n)y_n - \alpha_n Ty_n||$, $n = 0,1,\ldots,$ and suppose that $\lim_{n\to\infty} \epsilon_n = 0$. Then, we shall prove that $\lim_{n\to\infty} y_n = p$, by using (5) and the triangle inequality. Therefore,

$$\begin{aligned}
||y_{n+1} - p|| &\leq ||y_{n+1} - (1-\alpha_n)y_n - \alpha_n Ty_n|| + ||(1-\alpha_n)y_n + \alpha_n Ty_n - p|| \\
&= \epsilon_n + ||(1-\alpha_n)y_n + \alpha_n Ty_n - (1-\alpha_n+\alpha_n)p|| \\
&= ||(1-\alpha_n)(y_n - p) + \alpha_n(Ty_n - p)|| + \epsilon_n \\
&\leq (1-\alpha_n)||y_n - p|| + \alpha_n||Ty_n - p|| + \epsilon_n \\
&= (1-\alpha_n)||y_n - p|| + \alpha_n||Ty_n - Tp|| + \epsilon_n \\
&= (1-\alpha_n)||y_n - p|| + \alpha_n||Tp - Ty_n|| + \epsilon_n \\
&\leq (1-\alpha_n)||y_n - p|| + \alpha_n\Psi(||p - y_n||)e^{L||p-Tp||} + \epsilon_n \\
&= (1-\alpha_n)||y_n - p|| + \alpha_n\Psi(||y_n - p||) + \epsilon_n \\
&\leq (1-\alpha)||y_n - p|| + \alpha\Psi(||y_n - p||) + \epsilon_n. \qquad (11)
\end{aligned}$$

By applying Lemma 2 in (11), we obtain

$$\lim_{n\to\infty} y_n = p.$$

Conversely, let $\lim_{n\to\infty} y_n = p$. Then, by using (5) and the triangle inequality, we have

$$\begin{aligned}
\epsilon_n &= ||y_{n+1} - (1-\alpha_n)y_n - \alpha_n Ty_n|| \\
&\leq ||(y_{n+1} - p|| + ||p - (1-\alpha_n)y_n - \alpha_n Ty_n|| \\
&= ||y_{n+1} - p|| + ||(1-\alpha_n+\alpha_n)p - (1-\alpha_n)y_n - \alpha_n Ty_n|| \\
&= ||y_{n+1} - p|| + ||(1-\alpha_n)(p - y_n) + \alpha_n(p - Ty_n)|| \\
&\leq ||y_{n+1} - p|| + (1-\alpha_n)||y_n - p|| + \alpha_n\Psi(||p - y_n||)e^{L||p-Tp||} \\
&= ||y_{n+1} - p|| + (1-\alpha_n)||y_n - p|| + \alpha_n\Psi(||y_n - p||) \to 0 \\
&\leq ||y_{n+1} - p|| + (1-\alpha)||y_n - p|| + \alpha\Psi(||y_n - p||) \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Remark 10 Theorem 3 is a generalization of Theorem 3.2 of Imoru and Olatinwo [5], while Theorem 4 is a generalization of Theorem M of Imoru et al [6]. Moreover, each of both Theorem 3.2 of [5] and Theorem M of [6] is itself a generalization of Theorem 3 of Harder and Hicks [4] ,Theorem 2 of Rhoades [10, 11] and Theorem 3 of Berinde [1].

Remark 11 If in (4), $\Phi(u) = e^{Lu}$, $L \geq 0$, or, in (5), $\Psi(u) = au$, $a \in [0,1)$, $u \in \Re_+$, then we obtain the following contractive definition: there exist $a \in [0,1)$ and a constant $L \geq 0$ such that $\forall x, y \in X$,

$$d(Tx,Ty) \leq ad(x,y)e^{Ld(x,Tx)}. \tag{12}$$

By Remark 11, we obtain the following corollary to Theorems 1 and 2.

Corollary 1 *Let (X,d) be a complete metric space and $T : X \to X$ a selfmap of X satisfying (12). Suppose that T has a fixed point p. Let $x_o \in X$ and let*

$$x_{n+1} = f(T, x_n) = Tx_n, \quad n = 0, 1, 2, \ldots$$

be the Picard iteration. Then, the Picard iteration is T-stable.

In a similar manner, we obtain the following corollary to Theorems 3 and 4.

Corollary 2 *Let $(X, ||\cdot||)$ be a normed linear space and $T : X \to X$ a selfmap of X satisfying (12). Suppose that T has a fixed point p. Let $x_0 \in X$ and let*

$$x_{n+1} = f(T, x_n) = (1-\alpha_n)x_n + \alpha_n Tx_n, \quad \alpha_n \in [0,1],\ n = 0, 1, \ldots$$

be the Mann iteration process such that $0 < \alpha \leq \alpha_n$. Then, the Mann iteration process is T-stable.

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Variance Components and Nonlinearity*

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Abstract

Unknown parameters of the covariance matrix (variance components) of the observation vector in regression models are an unpleasant obstacle in a construction of the best estimator of the unknown parameters of the mean value of the observation vector. Estimators of variance componets must be utilized and then it is difficult to obtain the distribution of the estimators of the mean value parameters. The situation is more complicated in the case of nonlinearity of the regression model. The aim of the paper is to contribute to a solution of the mentioned problem.

Key words: Variance components; nonlinear regression model; linearization region; insensitiveness region.

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1 Introduction

The regression model is assumed to be of the form

$$\mathbf{Y} \sim_n \left(\mathbf{f}(\boldsymbol{\beta}), \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i \right),$$

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where $\mathbf{Y}$ is an n-dimensional random vector (observation vector) with the mean value equal to $\mathbf{f}(\boldsymbol{\beta}), \boldsymbol{\beta} \in R^k$ (k-dimensional Euclidean space) and the covariance matrix equal to $\sum_{i=1}^{p} \vartheta_i \mathbf{V}_i$. Here $\boldsymbol{\vartheta} = (\vartheta_1, \ldots, \vartheta_p)'$ is a p-dimensional vector of variance components and $\boldsymbol{\vartheta} \in \underline{\vartheta} \subset R^p$; $\underline{\vartheta}$ is an open set in R^p. The symmetric and positive semidefinite (p.s.d.) matrices $\mathbf{V}_1, \ldots, \mathbf{V}_p$ are given and all variance components are positive.

The problem is to find a decision whether the model can be linearized (with respect to $\boldsymbol{\beta}$) and estimators of the variance components ($\boldsymbol{\vartheta}$) can be used instead of the true values in estimation of $\boldsymbol{\beta}$. One of the possible approaches is demonstrated in the case of the bias of the estimator of $\boldsymbol{\beta}$.

2 Preliminaries

In the following text it will be assumed that the model considered can be characterized with sufficient accuracy as

$$\mathbf{Y} - \mathbf{f}_0 \sim N_n\left(\mathbf{F}\delta\boldsymbol{\beta} + \frac{1}{2}\boldsymbol{\kappa}(\delta\boldsymbol{\beta}), \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i\right) \tag{1}$$

where

$$\mathbf{f}_0 = \mathbf{f}(\boldsymbol{\beta}_0), \quad \mathbf{F} = \frac{\partial \mathbf{f}(\mathbf{u})}{\partial \mathbf{u}'}\Big|_{u=\beta_0}, \quad \boldsymbol{\kappa}(\delta\boldsymbol{\beta}) = [\kappa_1(\delta\boldsymbol{\beta}), \ldots, \kappa_n(\delta\boldsymbol{\beta})]',$$

$$\kappa_i(\delta\boldsymbol{\beta}) = \delta\boldsymbol{\beta}' \frac{\partial^2 f_i(\mathbf{u})}{\partial \mathbf{u} \partial \mathbf{u}'}\Big|_{u=\beta_0} \delta\boldsymbol{\beta}, \quad i = 1, \ldots, n,$$

and the vector $\boldsymbol{\beta}_0$ is as near as possible to the true value $\boldsymbol{\beta}^*$ of the parameter $\boldsymbol{\beta}$.

The linear version of the model considered is

$$\mathbf{Y} - \mathbf{f}_0 \sim N_n\left(\mathbf{F}\delta\boldsymbol{\beta}, \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i\right), \delta\boldsymbol{\beta} \in R^k, \boldsymbol{\vartheta} \in \underline{\vartheta}. \tag{2}$$

The regularity of the model will be assumed in the following consideration, i.e., the rank of the matrix $\mathbf{F}$ is $r(\mathbf{F}) = k < n$, and $\forall\{\boldsymbol{\vartheta} \in \underline{\vartheta}\}\boldsymbol{\Sigma}(\boldsymbol{\vartheta}) = \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i$ is positive definite (p.d.).

Lemma 2.1 *In the model (2) the $\boldsymbol{\vartheta}_0$-LBLUE (locally best linear unbiased estimator) of the parameter $\boldsymbol{\beta}$ is $\hat{\boldsymbol{\beta}} = \boldsymbol{\beta}_0 + \widehat{\delta\boldsymbol{\beta}}$, where*

$$\begin{aligned}\widehat{\delta\boldsymbol{\beta}} &= [\mathbf{F}'(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{F}]^{-1}\mathbf{F}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)(\mathbf{Y} - \mathbf{f}_0) \\ &\sim N_k\left(\delta\boldsymbol{\beta}, [\mathbf{F}'(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{F}]^{-1}\boldsymbol{\Sigma}(\boldsymbol{\vartheta}^*)[\mathbf{F}'(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{F}]^{-1}\right).\end{aligned}$$

Here $\boldsymbol{\vartheta}^*$ is the actual value of the vector parameter $\boldsymbol{\vartheta}$.

Proof is well known and therefore it is omitted.

The notation $\mathbf{S}_A$ ($\mathbf{A}$ is any $n \times n$ matrix) means the matrix with the (i,j)-th entry equal to

$$\{\mathbf{S}_A\}_{i,j} = \mathrm{Tr}(\mathbf{V}_i \mathbf{A} \mathbf{V}_j \mathbf{A}), \quad i,j = 1,\dots,p.$$

Further $(\mathbf{M}_F \boldsymbol{\Sigma}_0 \mathbf{M}_F)^+$ is the Moore–Penrose generalized inverse of the matrix $\mathbf{M}_F \boldsymbol{\Sigma}_0 \mathbf{M}_F$, $\mathbf{M}_F = \mathbf{I} - \mathbf{P}_F = \mathbf{F}\mathbf{F}^+$ (in more detail cf. [7]).

Lemma 2.2 *Let in the model (2) the matrix* $\mathbf{S}_{(M_F \Sigma_0 M_F)^+}$ *be regular* ($\boldsymbol{\Sigma}_0 = \sum_{i=1}^p \vartheta_i^{(0)} \mathbf{V}_i$, $\boldsymbol{\vartheta}^{(0)}$ *is the value of the parameter* $\boldsymbol{\vartheta}$ *as near as possible to the actual value* $\boldsymbol{\vartheta}^*$*). Then the* $\boldsymbol{\vartheta}_0$*-MINQUE (minimin norm quadratic unbiased estimator; in more detail cf. [8]) of the vector* $\boldsymbol{\vartheta}$ *is*

$$\hat{\boldsymbol{\vartheta}} = \mathbf{S}^{-1}_{(M_F \Sigma_0 M_F)^+} \begin{pmatrix} (\mathbf{Y} - \mathbf{f}_0)'(\mathbf{M}_F \boldsymbol{\Sigma}_0 \mathbf{M}_F)^+ \mathbf{V}_1 (\mathbf{M}_F \boldsymbol{\Sigma}_0 \mathbf{M}_F)^+ (\mathbf{Y} - \mathbf{f}_0) \\ \vdots \\ (\mathbf{Y} - \mathbf{f}_0)'(\mathbf{M}_F \boldsymbol{\Sigma}_0 \mathbf{M}_F)^+ \mathbf{V}_p (\mathbf{M}_F \boldsymbol{\Sigma}_0 \mathbf{M}_F)^+ (\mathbf{Y} - \mathbf{f}_0) \end{pmatrix}.$$

In the case of normality the variance matrix of this estimator is $\mathrm{Var}_{\vartheta_0}(\hat{\boldsymbol{\vartheta}}) = 2\mathbf{S}^{-1}_{(M_F \Sigma_0 M_F)^+}$.

Proof Cf. [8].

3 Influence of nonlinearity on the estimator of $\boldsymbol{\vartheta}$

Lemma 3.1 *In the model (1) the bias of the estimator from Lemma 2.2 at the point* $\boldsymbol{\beta}_0$ *is*

$$E_{\beta_0,\vartheta}(\hat{\boldsymbol{\vartheta}}) - \boldsymbol{\vartheta} = \frac{1}{4} \mathbf{S}^{-1}_{(M_X \Sigma_0 M_F)^+} \begin{pmatrix} \boldsymbol{\kappa}'(\delta\boldsymbol{\beta})(\mathbf{M}_F \boldsymbol{\Sigma}_0 \mathbf{M}_F)^+ \mathbf{V}_1 (\mathbf{M}_F \boldsymbol{\Sigma}_0 \mathbf{M}_F)^+ \boldsymbol{\kappa}(\delta\boldsymbol{\beta}) \\ \vdots \\ \boldsymbol{\kappa}'(\delta\boldsymbol{\beta})(\mathbf{M}_F \boldsymbol{\Sigma}_0 \mathbf{M}_F)^+ \mathbf{V}_p (\mathbf{M}_F \boldsymbol{\Sigma}_0 \mathbf{M}_F)^+ \boldsymbol{\kappa}(\delta\boldsymbol{\beta}) \end{pmatrix}.$$

Proof It is valid

$$\begin{aligned} & E_{\beta_0,\vartheta}\Big[(\mathbf{Y} - \mathbf{f}_0)'(\mathbf{M}_F \boldsymbol{\Sigma}_0 \mathbf{M}_F)^+ \mathbf{V}_j (\mathbf{M}_F \boldsymbol{\Sigma}_0 \mathbf{M}_F)^+ (\mathbf{Y} - \mathbf{f}_0)\Big] \\ &= E_{\beta_0,\vartheta}(\mathbf{Y} - \mathbf{f}_0)'(\mathbf{M}_F \boldsymbol{\Sigma}_0 \mathbf{M}_F)^+ \mathbf{V}_j (\mathbf{M}_F \boldsymbol{\Sigma}_0 \mathbf{M}_F)^+ E_{\beta_0,\vartheta}(\mathbf{Y} - \mathbf{f}_0) \\ &\quad + \mathrm{Tr}\left[(\mathbf{M}_F \boldsymbol{\Sigma}_0 \mathbf{M}_F)^+ \mathbf{V}_j (\mathbf{M}_F \boldsymbol{\Sigma}_0 \mathbf{M}_F)^+ \boldsymbol{\Sigma}(\boldsymbol{\vartheta})\right] \end{aligned}$$

Now it is sufficient to use the equalities

$$E_{\beta_0,\vartheta}(\mathbf{Y} - \mathbf{f}_0) = \mathbf{F}\delta\boldsymbol{\beta} + \frac{1}{2}\boldsymbol{\kappa}(\delta\boldsymbol{\beta})$$

and

$$\mathrm{Tr}\left[(\mathbf{M}_F \boldsymbol{\Sigma}_0 \mathbf{M}_F)^+ \mathbf{V}_j (\mathbf{M}_F \boldsymbol{\Sigma}_0 \mathbf{M}_F)^+ \boldsymbol{\Sigma}(\boldsymbol{\vartheta})\right] = \left\{\mathbf{S}_{(M_F \Sigma_0 M_F)^+}\right\}_{j,\cdot} \boldsymbol{\vartheta}. \qquad \square$$

Let the Bates and Watts intrinsic measure of nonlinearity [1] at the point $(\boldsymbol{\beta}_0, \boldsymbol{\vartheta}_0)$ be denoted as $K^{(int)}_{\vartheta_0}(\boldsymbol{\beta}_0)$,

$$K^{(int)}_{\vartheta_0}(\boldsymbol{\beta}_0) = \sup\left\{ \frac{\sqrt{\boldsymbol{\kappa}'(\delta\boldsymbol{\beta})\boldsymbol{\Sigma}_0^{-1}\mathbf{M}_F^{\Sigma_0^{-1}}\boldsymbol{\kappa}(\delta\boldsymbol{\beta})}}{\delta\boldsymbol{\beta}'\mathbf{F}'\boldsymbol{\Sigma}_0^{-1}\mathbf{F}\delta\boldsymbol{\beta}} : \delta\boldsymbol{\beta} \in R^k \right\}.$$

Theorem 3.2 *Let* $\mathbf{C}_0 = \mathbf{F}'\boldsymbol{\Sigma}_0^{-1}\mathbf{F}$. *If*

$$\delta\boldsymbol{\beta}'\mathbf{C}_0\delta\boldsymbol{\beta} \le \frac{2\varepsilon}{K^{(int)}_{\vartheta_0}(\boldsymbol{\beta}_0)},$$

then

$$\forall\{i = 1, \dots, p\}\ |E_{\beta_0,\vartheta}(\hat{\vartheta}_i) - \vartheta_i| \le \sum_{i=1}^{p} |k_{i,j}|\varepsilon^2,$$

where

$$\mathbf{k}_i' = (k_{i,1}, \dots, k_{i,p}) = \left\{\mathbf{S}^{-1}_{(M_F\Sigma_0 M_F)^+}\right\}_{i,\cdot} [\mathrm{Diag}(\boldsymbol{\vartheta}_0)]^{-1}, \quad i = 1, \dots, p.$$

Proof Let $\hat{\boldsymbol{\zeta}} = (\hat{\zeta}_1, \dots, \hat{\zeta}_p)'$, where

$$\hat{\zeta}_i = (\mathbf{Y} - \mathbf{f}_0)'(\mathbf{M}_F\boldsymbol{\Sigma}_0\mathbf{M}_F)^+\mathbf{V}_i(\mathbf{M}_F\boldsymbol{\Sigma}_0\mathbf{M}_F)^+(\mathbf{Y} - \mathbf{f}_0), \quad i = 1, \dots, p.$$

Then, with respect to Lemma 3.1

$$\begin{aligned}
&E_{\beta_0,\vartheta}(\hat{\boldsymbol{\vartheta}}) - \boldsymbol{\vartheta} = \\
&= \mathbf{S}^{-1}_{(M_F\Sigma_0 M_F)^+}\left[E_{\beta_0,\vartheta}(\hat{\boldsymbol{\zeta}}) - \mathbf{S}_{(M_F\Sigma_0 M_F)^+}\boldsymbol{\vartheta}\right] \\
&= \mathbf{S}^{-1}_{(M_F\Sigma_0 M_F)^+}\frac{1}{4}\begin{pmatrix} \boldsymbol{\kappa}'(\delta\boldsymbol{\beta})(\mathbf{M}_F\boldsymbol{\Sigma}_0\mathbf{M}_F)^+\mathbf{V}_1(\mathbf{M}_F\boldsymbol{\Sigma}_0\mathbf{M}_F)^+\boldsymbol{\kappa}(\delta\boldsymbol{\beta}) \\ \vdots \\ \boldsymbol{\kappa}'(\delta\boldsymbol{\beta})(\mathbf{M}_F\boldsymbol{\Sigma}_0\mathbf{M}_F)^+\mathbf{V}_p(\mathbf{M}_F\boldsymbol{\Sigma}_0\mathbf{M}_F)^+\boldsymbol{\kappa}(\delta\boldsymbol{\beta}) \end{pmatrix} \\
&= \mathbf{S}^{-1}_{(M_F\Sigma_0 M_F)^+}\left[\mathrm{Diag}(\boldsymbol{\vartheta}_0)\right]^{-1}\mathrm{Diag}(\boldsymbol{\vartheta}_0) \\
&\quad \times\frac{1}{4}\begin{pmatrix} \boldsymbol{\kappa}'(\delta\boldsymbol{\beta})(\mathbf{M}_F\boldsymbol{\Sigma}_0\mathbf{M}_F)^+\mathbf{V}_1(\mathbf{M}_F\boldsymbol{\Sigma}_0\mathbf{M}_F)^+\boldsymbol{\kappa}(\delta\boldsymbol{\beta}) \\ \vdots \\ \boldsymbol{\kappa}'(\delta\boldsymbol{\beta})(\mathbf{M}_F\boldsymbol{\Sigma}_0\mathbf{M}_F)^+\mathbf{V}_p(\mathbf{M}_F\boldsymbol{\Sigma}_0\mathbf{M}_F)^+\boldsymbol{\kappa}(\delta\boldsymbol{\beta}) \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{k}_1' \\ \vdots \\ \mathbf{k}_p' \end{pmatrix}\frac{1}{4}\begin{pmatrix} \boldsymbol{\kappa}'(\delta\boldsymbol{\beta})(\mathbf{M}_F\boldsymbol{\Sigma}_0\mathbf{M}_F)^+\vartheta_1^{(0)}\mathbf{V}_1(\mathbf{M}_F\boldsymbol{\Sigma}_0\mathbf{M}_F)^+\boldsymbol{\kappa}(\delta\boldsymbol{\beta}) \\ \vdots \\ \boldsymbol{\kappa}'(\delta\boldsymbol{\beta})(\mathbf{M}_F\boldsymbol{\Sigma}_0\mathbf{M}_F)^+\vartheta_p^{(0)}\mathbf{V}_p(\mathbf{M}_F\boldsymbol{\Sigma}_0\mathbf{M}_F)^+\boldsymbol{\kappa}(\delta\boldsymbol{\beta}) \end{pmatrix}.
\end{aligned}$$

The inclusions $\mathcal{M}(\mathbf{V}_i) = \{\mathbf{V}_i\mathbf{u} : \mathbf{u} \in R^n\} \subset \mathcal{M}(\boldsymbol{\Sigma}_0)$, $i = 1, \dots, p$, are a consequnce of the assumption that the matrices $\mathbf{V}_1, \dots, \mathbf{V}_p$ are p.s.d. and $\vartheta_i > 0$, $i = 1, \dots, p$. These inclusions imply

$$\begin{gathered}
\boldsymbol{\kappa}'(\delta\boldsymbol{\beta})(\mathbf{M}_F\boldsymbol{\Sigma}_0\mathbf{M}_F)^+\vartheta_i^{(0)}\mathbf{V}_i(\mathbf{M}_F\boldsymbol{\Sigma}_0\mathbf{M}_F)^+\boldsymbol{\kappa}(\delta\boldsymbol{\beta}) \\
\le \boldsymbol{\kappa}'(\delta\boldsymbol{\beta})(\mathbf{M}_F\boldsymbol{\Sigma}_0\mathbf{M}_F)^+\boldsymbol{\Sigma}_0(\mathbf{M}_F\boldsymbol{\Sigma}_0\mathbf{M}_F)^+\boldsymbol{\kappa}(\delta\boldsymbol{\beta}) = \boldsymbol{\kappa}'(\delta\boldsymbol{\beta})(\mathbf{M}_F\boldsymbol{\Sigma}_0\mathbf{M}_F)^+\boldsymbol{\kappa}(\delta\boldsymbol{\beta}).
\end{gathered}$$

Thus we obtain

$$|E_{\beta_0,\vartheta}(\hat{\vartheta}_i) - \vartheta_i| = \frac{1}{4}\sum_{j=1}^{p} k_{i,j}\boldsymbol{\kappa}'(\delta\boldsymbol{\beta})(\mathbf{M}_F\boldsymbol{\Sigma}_0\mathbf{M}_F)^+\vartheta_j^{(0)}\mathbf{V}_j(\mathbf{M}_F\boldsymbol{\Sigma}_0\mathbf{M}_F)^+\boldsymbol{\kappa}(\delta\boldsymbol{\beta})$$
$$\leq \frac{1}{4}\sum_{j=1}^{p}|k_{i,j}|\boldsymbol{\kappa}'(\delta\boldsymbol{\beta})(\mathbf{M}_F\boldsymbol{\Sigma}_0\mathbf{M}_F)^+\boldsymbol{\kappa}(\delta\boldsymbol{\beta}).$$

Now the definition of $K_{\vartheta_0}^{(int)}(\boldsymbol{\beta}_0)$ can be used and thus

$$\boldsymbol{\kappa}'(\delta\boldsymbol{\beta})(\mathbf{M}_F\boldsymbol{\Sigma}_0\mathbf{M}_F)^+\boldsymbol{\kappa}(\delta\boldsymbol{\beta}) \leq \left(K_{\vartheta_0}^{(int)}(\boldsymbol{\beta}_0)\right)^2(\delta\boldsymbol{\beta}'\mathbf{C}_0\delta\boldsymbol{\beta})^2.$$

If

$$\delta\boldsymbol{\beta}'\mathbf{C}_0\delta\boldsymbol{\beta} \leq \frac{2\varepsilon}{K_{\vartheta_0}^{(int)}(\boldsymbol{\beta}_0)},$$

then

$$\boldsymbol{\kappa}'(\delta\boldsymbol{\beta})(\mathbf{M}_F\boldsymbol{\Sigma}_0\mathbf{M}_F)^+\boldsymbol{\kappa}(\delta\boldsymbol{\beta}) \leq 4\varepsilon^2$$

and also

$$|E_{\beta_0,\vartheta}(\hat{\vartheta}_i) - \vartheta_i| \leq \frac{1}{4}\sum_{j=1}^{p}|k_{i,j}|\boldsymbol{\kappa}'(\delta\boldsymbol{\beta})(\mathbf{M}_F\boldsymbol{\Sigma}_0\mathbf{M}_F)^+\boldsymbol{\kappa}(\delta\boldsymbol{\beta}) \leq \sum_{j=1}^{p}|k_{i,j}|\varepsilon^2. \quad \square$$

4 Linearization region

In the case of the model (2) when variance components are known, then the BLUE of $\boldsymbol{\beta}$ is

$$\hat{\boldsymbol{\beta}} = [\mathbf{F}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta})\mathbf{F}]^{-1}\mathbf{F}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta})\mathbf{Y}.$$

This estimator is biased in the model (1) and

$$\mathbf{b} = E_\beta(\hat{\boldsymbol{\beta}}) - \boldsymbol{\beta} = \frac{1}{2}[\mathbf{F}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta})\mathbf{F}]^{-1}\mathbf{F}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta})\boldsymbol{\kappa}(\delta\boldsymbol{\beta}).$$

Let the Bates and Watts parametric curvature at the point $(\boldsymbol{\beta}_0, \boldsymbol{\vartheta}_0)$ be denoted as $K_{\vartheta_0}^{(par)}(\boldsymbol{\beta}_0)$

$$K_{\vartheta_0}^{(par)}(\boldsymbol{\beta}_0) = \sup\left\{\frac{\sqrt{\boldsymbol{\kappa}'(\delta\boldsymbol{\beta})\boldsymbol{\Sigma}_0^{-1}\mathbf{P}_F^{\Sigma_0^{-1}}\boldsymbol{\kappa}(\delta\boldsymbol{\beta})}}{\delta\boldsymbol{\beta}'\mathbf{F}'\boldsymbol{\Sigma}_0^{-1}\mathbf{F}\delta\boldsymbol{\beta}} : \delta\boldsymbol{\beta} \in R^k\right\}.$$

Lemma 4.1 *Let in the model (1)*

$$\delta\boldsymbol{\beta}'\mathbf{F}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{F}\delta\boldsymbol{\beta} \leq \frac{2\varepsilon}{K_{\vartheta_0}^{(par)}(\boldsymbol{\beta}_0)}.$$

Then

$$\forall\{\mathbf{h} \in R^k\}|\mathbf{h}'\mathbf{b}| \leq \varepsilon\sqrt{\mathbf{h}'[\mathbf{F}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{F}]^{-1}\mathbf{h}}.$$

Proof Cf. in [4] and [6].

Remark 4.2 Theorem 3.2 and Lemma 4.1 show that the regions of linearization for $\boldsymbol{\vartheta}$

$$\mathcal{L}_\vartheta = \left\{ \delta\boldsymbol{\beta} : \delta\boldsymbol{\beta}'\mathbf{F}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{F}\delta\boldsymbol{\beta} \le \frac{2\varepsilon}{K_{\vartheta_0}^{(int)}(\boldsymbol{\beta}_0)} \right\}$$

and for the bias $\mathbf{b}$

$$\mathcal{L}_b = \left\{ \delta\boldsymbol{\beta} : \delta\boldsymbol{\beta}'\mathbf{F}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{F}\delta\boldsymbol{\beta} \le \frac{2\varepsilon}{K_{\vartheta_0}^{(par)}(\boldsymbol{\beta}_0)} \right\}$$

have the same shape, i.e. we have to use the smaller of them. Usually $\mathcal{L}_b \subset \mathcal{L}_\vartheta$.

The necessary condition for efficient utilization of Theorem 3.2. and Lemma 4.1 is $\delta\boldsymbol{\beta}^* \in \mathcal{L}_b \cap \mathcal{L}_\vartheta$ and at the same time the difference $\boldsymbol{\vartheta}^* - \boldsymbol{\vartheta}_0$ must be in so called nonsensitivenes region which is in more detail described in the following section.

5 Nonsensitiveness region

How the small shift $\delta\boldsymbol{\vartheta}$ of the parameter $\boldsymbol{\vartheta}$ can change the statistical properties of the estimator $\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta})$ is given in the following statement.

Lemma 5.1 *Let*

$$\begin{aligned}
\mathbf{h}'\hat{\boldsymbol{\beta}}(\mathbf{Y}, \boldsymbol{\vartheta}_0 + \delta\boldsymbol{\vartheta}) &= \mathbf{h}'[\mathbf{F}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0 + \delta\boldsymbol{\vartheta})\mathbf{F}]^{-1}\mathbf{F}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0 + \delta\boldsymbol{\vartheta})\mathbf{Y}, \\
\mathbf{h}'\hat{\boldsymbol{\beta}}(\mathbf{Y}, \boldsymbol{\vartheta}_0) &= \mathbf{h}'[\mathbf{F}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{F}]^{-1}\mathbf{F}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{Y}, \\
\mathbf{v} &= \mathbf{Y} - \mathbf{F}\hat{\boldsymbol{\beta}}(\mathbf{Y}, \boldsymbol{\vartheta}_0).
\end{aligned}$$

Then

(i) $\mathbf{h}'\hat{\boldsymbol{\beta}}(\mathbf{Y}, \boldsymbol{\vartheta}_0 + \delta\boldsymbol{\vartheta}) = \mathbf{h}'\hat{\boldsymbol{\beta}}(\mathbf{Y}, \boldsymbol{\vartheta}_0) - \mathbf{L}_h'\boldsymbol{\Sigma}(\delta\boldsymbol{\vartheta})\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{v}$,
where $\boldsymbol{\Sigma}(\delta\boldsymbol{\vartheta}) = \sum_{i=1}^p \delta\vartheta_i\mathbf{V}_i$ *and* $\mathbf{L}_h' = \mathbf{h}'[\mathbf{F}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{F}]^{-1}\mathbf{F}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)$.

(ii) $E_\beta(\mathbf{L}_h'\boldsymbol{\Sigma}(\delta\boldsymbol{\vartheta})\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{v}) = \mathbf{0}$.

(iii) $\operatorname{cov}_{\vartheta_0}\left(\mathbf{L}_h'\boldsymbol{\Sigma}(\delta\boldsymbol{\vartheta})\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{v}, \hat{\boldsymbol{\beta}}(\mathbf{Y}, \boldsymbol{\vartheta}_0)\right) = \mathbf{0}$.

Proof Cf. [2] and [3].

Corollary 5.2 *Let*

$$\mathbf{W}_h = \begin{pmatrix} \mathbf{L}_h'\mathbf{V}_1 \\ \vdots \\ \mathbf{L}_h'\mathbf{V}_p \end{pmatrix} [\mathbf{M}_F(\boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0)\mathbf{M}_F]^+(\mathbf{V}_1\mathbf{L}_h, \dots, \mathbf{V}_p\mathbf{L}_h).$$

Then

$$\mathrm{Var}_{\vartheta_0}[\mathbf{h}'\hat{\boldsymbol{\beta}}(\mathbf{Y}, \boldsymbol{\vartheta}_0 + \delta\boldsymbol{\vartheta})] = \mathrm{Var}_{\vartheta_0}[\mathbf{h}'\hat{\boldsymbol{\beta}}(\mathbf{Y}, \boldsymbol{\vartheta}_0)] + \delta\boldsymbol{\vartheta}'\mathbf{W}_h\delta\boldsymbol{\vartheta}.$$

If an experimenter can admit

$$\sqrt{\mathrm{Var}_{\vartheta_0}[\mathbf{h}'\hat{\boldsymbol{\beta}}(\mathbf{Y}, \boldsymbol{\vartheta}_0)] + \delta\boldsymbol{\vartheta}'\mathbf{W}_h\delta\boldsymbol{\vartheta}} \leq (1+\varepsilon)\sqrt{\mathrm{Var}_{\vartheta_0}[\mathbf{h}'\hat{\boldsymbol{\beta}}(\mathbf{Y}, \boldsymbol{\vartheta}_0)]},$$

then $\delta\boldsymbol{\vartheta}^*$ must be in the region

$$\mathcal{N}_h = \left\{\delta\boldsymbol{\vartheta} : \delta\boldsymbol{\vartheta}'\mathbf{W}_h\delta\boldsymbol{\vartheta} \leq 2\varepsilon\mathbf{h}'\left[\mathbf{F}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{F}\right]^{-1}\mathbf{h}\right\}.$$

In order to recognize whether $\delta\boldsymbol{\beta}^*$ and $\delta\boldsymbol{\vartheta}^*$ are in the regions $\mathcal{L}_b \cap \mathcal{L}_\vartheta$ and $\mathcal{N}_h$, respectively, we must have some information an an accuracy of the estimators $\hat{\boldsymbol{\beta}}$ and $\hat{\boldsymbol{\vartheta}}$.

The first orientation on the confidence region of the parameter $\boldsymbol{\beta}$ is the set

$$\mathcal{E}_\beta = \left\{\delta\boldsymbol{\beta} : (\delta\boldsymbol{\beta} - \widehat{\delta\boldsymbol{\beta}})'\mathbf{F}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{F}(\delta\boldsymbol{\beta} - \widehat{\delta\boldsymbol{\beta}}) \leq \chi^2_k(0, 1-\alpha)\right\},$$

where $\chi^2_k(0, 1-\alpha)$ is the $(1-\alpha)$-quantile of the central chi-square distribution with k degrees of freedom.

Unfortunately the confidence region for the parameter $\boldsymbol{\vartheta}$ is not known, however some information on it we can obtain by the help of the following lemma.

Lemma 5.3 *Let*

$$\mathbf{Y} \sim N_n(\mathbf{f}_0 + \mathbf{F}\delta\boldsymbol{\beta} + \frac{1}{2}\boldsymbol{\kappa}(\delta\boldsymbol{\beta}), \sum_{i=1}^{p}\vartheta_i\mathbf{V}_i).$$

Then

$$\delta\boldsymbol{\vartheta}'(2\mathbf{S}^{-1}_{(M_F\Sigma_0 M_F)^+})^{-1}\delta\boldsymbol{\vartheta} < t^2$$

$$\Rightarrow \forall\{i = 1, \dots, p\}\ |\vartheta_i| \leq t\sqrt{2\{\mathbf{S}^{-1}_{(M_F\Sigma_0 M_F)^+}\}_{i,i}}.$$

Proof It is a direct consequence of Theorem 2.2. in [5]

Remark 5.4 If the real number $t > 0$ is sufficiently large such that

$$|\hat{\vartheta}_i - \vartheta^*| < t\sqrt{2\{\mathbf{S}^{-1}_{(M_F\Sigma_0 M_F)^+}\}_{i,i}}, \quad i = 1, \dots, p,$$

occur with certainty (with sufficiently high probability), then we can be practically sure that the actual value $\boldsymbol{\vartheta}^*$ of the vector $\boldsymbol{\vartheta}$ is in the domain

$$\mathcal{K}_\vartheta = \left\{\delta\boldsymbol{\vartheta} : (\delta\boldsymbol{\vartheta} - \widehat{\delta\boldsymbol{\vartheta}})'\mathbf{S}_{(M_F\Sigma_0 M_F)^+}(\delta\boldsymbol{\vartheta} - \widehat{\delta\boldsymbol{\vartheta}}) < 2t^2\right\}.$$

6 Inference on linearization

A comparison of the sets $\mathcal{E}_\beta, \mathcal{K}_\vartheta, \mathcal{N}_h, \mathcal{L}_b, \mathcal{L}_\vartheta$ leads to a decision whether the considered model with unknown variance components can be linearized. In the first step we shall take into account the following lemma.

Lemma 6.1 *Let $\boldsymbol{\vartheta}_0$ be given. Then for any $\tau > 0$ the notation $\boldsymbol{\vartheta}_\tau$ means $\tau\boldsymbol{\vartheta}_0$.*

(i) $\mathbf{k}_i'(\tau\boldsymbol{\vartheta}_0) = \left\{\mathbf{S}^{-1}_{(M_F\Sigma_{\tau\vartheta_0}M_F)^+}\right\}_{i,\cdot}[\mathrm{Diag}(\tau\boldsymbol{\vartheta}_0)]^{-1} = \tau\mathbf{k}_i'(\boldsymbol{\vartheta}_0)$.

(ii) $\mathbf{S}_{(M_F\Sigma_{\tau\vartheta_0}M_F)^+} = \tau^2\mathbf{S}_{(M_F\Sigma_{\vartheta_0}M_F)^+}$.

(iii) $K^{(int)}_{\tau\vartheta_0}(\boldsymbol{\beta}_0) = \sqrt{\tau}K^{(int)}_{\vartheta_0}(\boldsymbol{\beta}_0)$.

(iv) $\mathbf{W}_h(\tau\boldsymbol{\vartheta}_0) = \frac{1}{\tau}\mathbf{W}_h(\boldsymbol{\vartheta}_0)$.

Proof The statements are direct consequences of definitions.

Corollary 6.2 *If*

$$\sum_{j=1}^n |k_{i,j}(\boldsymbol{\vartheta}_0)|\varepsilon_1^2 \le \varepsilon_2\sqrt{\left\{2\mathbf{S}^{-1}_{M_F\Sigma_{\vartheta_0}M_F)^+}\right\}_{i,i}},$$

then

$$\forall\{\tau > 0\} \sum_{j=1}^n |k_{i,j}(\tau\boldsymbol{\vartheta}_0)|\varepsilon_1^2 \le \varepsilon_2\sqrt{\left\{2\mathbf{S}^{-1}_{M_F\Sigma_{\tau\vartheta_0}M_F)^+}\right\}_{i,i}}$$

(consequence of Lemma 6.1 (i) and (ii)).

$$\frac{1}{\tau}\delta\boldsymbol{\vartheta}'\mathbf{W}_h(\boldsymbol{\vartheta}_0)\delta\boldsymbol{\vartheta} = \delta\boldsymbol{\vartheta}'\mathbf{W}_h(\tau\boldsymbol{\vartheta}_0)\delta\boldsymbol{\vartheta} \le 2\varepsilon_3\,\mathrm{Var}_{\tau\vartheta_0}[\mathbf{h}'\hat{\boldsymbol{\beta}}(\tau\boldsymbol{\vartheta}_0)]$$
$$= \tau 2\varepsilon_3\,\mathrm{Var}_{\vartheta_0}[\mathbf{h}'\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0)]$$

(it is to be remarked that $\hat{\boldsymbol{\beta}}(\tau\boldsymbol{\vartheta}_0) = \hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0)$). The last inequality can be interpreted as follows. If the value $\boldsymbol{\vartheta}_0$ is changed into $\tau\boldsymbol{\vartheta}_0$, then the admissible shift $\delta\boldsymbol{\vartheta}$ is changed into the shif $\sqrt{\tau}\delta\boldsymbol{\vartheta}$.

Now the sequence of the steps necessary to make a decision can be described.

(i) When the values $\boldsymbol{\vartheta}_0$ and ε_2 are chosen the value ε_1 is determined in such a way that

$$\sum_{j=1}^n |k_{i,j}(\boldsymbol{\vartheta}_0)|\varepsilon_1^2 \le \varepsilon_2\sqrt{\{2\mathbf{S}^{-1}_{(M_F\Sigma_{\vartheta_0}M_F)^+}\}_{i,i}}, \quad i = 1,\dots,p,$$

(it implies $|E_{\vartheta_0}(\hat{\vartheta}_i) - \vartheta_i| \le \varepsilon_2\sqrt{\mathrm{Var}_{\vartheta_0}(\hat{\vartheta}_i)}$, $i = 1,\dots,p$, i.e. biases caused by nonlinearity can be neglected). Thus we determined the region $\mathcal{L}_{\vartheta_0}$, i.e.

$$\mathcal{L}_{\vartheta_0} = \left\{\delta\boldsymbol{\beta} : \delta\boldsymbol{\beta}'\mathbf{F}'\boldsymbol{\Sigma}^{-1}_{\vartheta_0}\mathbf{F}\delta\boldsymbol{\beta} \le \frac{2\varepsilon_1}{K^{(int)}_{\vartheta_0}(\boldsymbol{\beta}_0)}\right\}.$$

(ii) To choose the value ε_3 and to determine the set

$$\mathcal{N}_h = \left\{ \delta\boldsymbol{\vartheta} : \delta\boldsymbol{\vartheta}'\mathbf{W}_h(\boldsymbol{\vartheta}_0)\delta\boldsymbol{\vartheta} \leq 2\varepsilon_3 \operatorname{Var}_{\vartheta_0}\left[\mathbf{h}'\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0)\right]\right\}.$$

Shifts $\delta\boldsymbol{\vartheta}$ inside the set $\mathcal{N}_h$ does not enlarge the standard deviation of the estimator $\mathbf{h}'\widehat{\delta\boldsymbol{\beta}}$ more than $\varepsilon_3\sqrt{\operatorname{Var}_{\vartheta_0}(\mathbf{h}'\widehat{\boldsymbol{\beta}})}$.

(iii) To check the inclusions $\mathcal{E}_\beta \subset \mathcal{L}_b \cap \mathcal{L}_\vartheta$ and

$$\mathcal{K}_\vartheta = \left\{\mathbf{u} : \mathbf{u}'\mathbf{S}_{(M_F\Sigma_{\vartheta_0}M_F)^+}\mathbf{u}/2 \leq t^2\right\} \subset \mathcal{N}_h.$$

If these inclusions are satisfied (the actual value $\delta\boldsymbol{\beta}^*$ of $\delta\boldsymbol{\beta}$ is suffiently small for the bias of the estimator $\hat{\boldsymbol{\vartheta}}$ and the actual $\delta\boldsymbol{\vartheta}^*$ is with high probability in the nonsensitiveness region), then the model with the estimated variance components can be linearized and the estimates $\hat{\boldsymbol{\vartheta}}$ can be used for the estimation of $\boldsymbol{\beta}$ without any essential deterioration of the statistical properties.

However if the last inclusion is not satisfied, then the model with unknown variance components cannot be linearized and it would be necessary to prepare another experiment in order to make the estimators of $\boldsymbol{\vartheta}$ more precise. In more detail it is shown in the next section.

7 Numerical example

Two points A and B with coordinates $(0,0)$ and $(0,800)$ are located in a plane. Third point P is determined by measurement of the angles $\angle BAP$ and $\angle PBA$, respectively, and distances AP and BP. The coordinates and distances are given in meters, angles are given in sexagesimal system. Measurement are stochastically independent, the variance in measurement of angles is $\sigma_\omega^2 = (10'')^2 = (10/206264.806\,\text{rad})^2 = (4.848 \times 10^{-5}\,\text{rad})^2$ and the variance in measurement distances is $\sigma_D^2 = (0.05\,\text{m})^2$. Each angle is measured $M(=2)$-times and each distance is measured $N(=2)$-times. The approximate value of the parameter $\boldsymbol{\beta}$ is

$$\boldsymbol{\beta}^{(0)} = \begin{pmatrix} 107.180 \\ 400.000 \end{pmatrix}.$$

Thus the quadratized version of the model can be written as

$$\begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \\ \mathbf{Y}_3 \\ \mathbf{Y}_4 \end{pmatrix} \sim N_{2M+2N}\left[\begin{pmatrix} \mathbf{f}_1^{(0)} \\ \mathbf{f}_2^{(0)} \\ \mathbf{f}_3^{(0)} \\ \mathbf{f}_4^{(0)} \end{pmatrix} + \begin{pmatrix} \mathbf{F}_1 \\ \mathbf{F}_2 \\ \mathbf{F}_3 \\ \mathbf{F}_4 \end{pmatrix}\delta\boldsymbol{\beta} + \frac{1}{2}\begin{pmatrix} \boldsymbol{\kappa}_1(\delta\boldsymbol{\beta}) \\ \boldsymbol{\kappa}_2(\delta\boldsymbol{\beta}) \\ \boldsymbol{\kappa}_3(\delta\boldsymbol{\beta}) \\ \boldsymbol{\kappa}_4(\delta\boldsymbol{\beta}) \end{pmatrix}, \sigma_\omega^2\mathbf{V}_1 + \sigma_D^2\mathbf{V}_2\right],$$

where

$$
\begin{aligned}
\mathbf{f}_1 &= \arctan\frac{\boldsymbol{\beta}_2}{\boldsymbol{\beta}_1}, \quad \mathbf{f}_2 = \arctan\frac{\boldsymbol{\beta}_2}{\boldsymbol{\beta}_1 - 800},\\
\mathbf{f}_3 &= \sqrt{(\boldsymbol{\beta}_1 - 800)^2 + \boldsymbol{\beta}_2^2}, \quad \mathbf{f}_4 = \sqrt{\boldsymbol{\beta}_1^2 + \boldsymbol{\beta}_2^2},\\
\mathbf{f}_1^{(0)} &= \mathbf{1}_M \alpha_{AP}^{(0)}, \quad \mathbf{f}_2^{(0)} = \mathbf{1}_M \alpha_{BP}^{(0)}, \quad \mathbf{f}_3^{(0)} = \mathbf{1}_N D_{BP}^{(0)}, \quad \mathbf{f}_4^{(0)} = \mathbf{1}_N D_{AP}^{(0)},\\
\alpha_{AP}^{(0)} &= 75^0, \quad \alpha_{BP}^{(0)} = 150^0, \quad D_{BP}^{(0)} = 800.000 \quad D_{AP}^{(0)} = 414.110,\\
\mathbf{F}_1 &= \mathbf{1}_M \times \mathbf{f}_1' = \mathbf{1}_M \otimes \left(\frac{-\sin(\alpha_{AP}^{(0)})}{D_{AP}^{(0)}}, \frac{\cos(\alpha_{AP}^{(0)})}{D_{AP}^{(0)}}\right),\\
\mathbf{F}_2 &= \mathbf{1}_M \otimes \mathbf{f}_2' = \mathbf{1}_M \otimes \left(-\frac{\sin(\alpha_{BP}^{(0)})}{D_{BP}^{(0)}}, \frac{\cos(\alpha_{BP}^{(0)})}{D_{BP}^{(0)}}\right),\\
\mathbf{F}_3 &= \mathbf{1}_N \otimes \mathbf{f}_3' = \mathbf{1}_N \otimes \left(\cos(\alpha_{BP}^{(0)}), \sin(\alpha_{BP}^{(0)})\right),\\
\mathbf{F}_4 &= \mathbf{1}_N \otimes \mathbf{f}_4' = \mathbf{1}_N \otimes \left(\cos(\alpha_{AP}^{(0)}), \sin(\alpha_{AP}^{(0)})\right),\\
\mathbf{f}_1' &= \left(-\frac{0.9659258}{414.110}, \frac{0.2588190}{414.110}\right), \quad \mathbf{f}_2' = \left(-\frac{0.5000000}{800.000}, -\frac{0.8660254}{800.000}\right),\\
\mathbf{f}_3' &= (-0.8660254, 0.5000000), \quad \mathbf{f}_4' = (0.2588190, 0.9659258),\\
\boldsymbol{\kappa}_1(\delta\boldsymbol{\beta}) &= \mathbf{1}_M \otimes \delta\boldsymbol{\beta}' \begin{pmatrix} \frac{2\beta_1^{(0)}\beta_2^{(0)}}{[(\beta_1^{(0)})^2+(\beta_2^{(0)})^2]^2}, & \frac{(\beta_2^{(0)})^2-(\beta_1^{(0)})^2}{[(\beta_1^{(0)})^2+(\beta_2^{(0)})^2]^2}\\ \frac{(\beta_2^{(0)})^2-(\beta_1^{(0)})^2}{[(\beta_1^{(0)})^2+(\beta_2^{(0)})^2]^2}, & -\frac{2\beta_1^{(0)}\beta_2^{(0)}}{[(\beta_1^{(0)})^2+(\beta_2^{(0)})^2]^2} \end{pmatrix} \delta\boldsymbol{\beta}'\\
\boldsymbol{\kappa}_2(\delta\boldsymbol{\beta}) &= \mathbf{1}_M \otimes \delta\boldsymbol{\beta}' \begin{pmatrix} \frac{2(\beta_1^{(0)}-800)\beta_2^{(0)}}{[(\beta_1^{(0)}-800)^2+(\beta_2^{(0)})^2]^2}, & \frac{(\beta_1^{(0)}-800)^2-(\beta_2^{(0)})^2}{[(\beta_1^{(0)}-800)^2+(\beta_2^{(0)})^2]^2}\\ \frac{(\beta_1^{(0)}-800)^2-(\beta_2^{(0)})^2}{[(\beta_1^{(0)}-800)^2+(\beta_2^{(0)})^2]^2}, & \frac{2(\beta_1^{(0)}-800)\beta_2^{(0)}}{[(\beta_1^{(0)}-800)^2+(\beta_2^{(0)})^2]^2} \end{pmatrix} \delta\boldsymbol{\beta}'\\
\boldsymbol{\kappa}_3(\delta\boldsymbol{\beta}) &= \mathbf{1}_N \otimes \delta\boldsymbol{\beta}' \begin{pmatrix} \frac{(\beta_2^{(0)})^2}{[(\beta_1^{(0)}-800)^2+(\beta_2^{(0)})^2]^{3/2}}, & -\frac{(\beta_1^{(0)}-800)\beta_2^{(0)}}{[(\beta_1^{(0)}-800)^2+(\beta_2^{(0)})^2]^{3/2}}\\ -\frac{(\beta_1^{(0)}-800)\beta_2^{(0)}}{[(\beta_1^{(0)}-800)^2+(\beta_2^{(0)})^2]^{3/2}}, & \frac{(\beta_1^{(0)}-800)^2}{[(\beta_1^{(0)}-800)^2+(\beta_2^{(0)})^2]^{3/2}} \end{pmatrix} \delta\boldsymbol{\beta}'\\
\boldsymbol{\kappa}_4(\delta\boldsymbol{\beta}) &= \mathbf{1}_N \otimes \delta\boldsymbol{\beta}' \begin{pmatrix} \frac{(\beta_2^{(0)})^2}{[(\beta_1^{(0)})^2+(\beta_2^{(0)})^2]^{3/2}}, & -\frac{\beta_1^{(0)}\beta_2^{(0)}}{[(\beta_1^{(0)})^2+(\beta_2^{(0)})^2]^{3/2}}\\ -\frac{\beta_1^{(0)}\beta_2^{(0)}}{[(\beta_1^{(0)})^2+(\beta_2^{(0)})^2]^{3/2}}, & \frac{(\beta_1^{(0)})^2}{[(\beta_1^{(0)})^2+(\beta_2^{(0)})^2]^{3/2}} \end{pmatrix} \delta\boldsymbol{\beta}',\\
\mathbf{V}_1 &= \begin{pmatrix} \mathbf{I}_{2M,2M}, & \mathbf{0}_{2M,2N}\\ \mathbf{0}_{2N,2M}, & \mathbf{0}_{2N,2N} \end{pmatrix}, \quad \mathbf{V}_2 = \begin{pmatrix} \mathbf{0}_{2M,2M}, & \mathbf{0}_{2M,2N}\\ \mathbf{0}_{2N,2M}, & \mathbf{I}_{2N,2N} \end{pmatrix}.
\end{aligned}
$$

Let

$$
c_{i,j} = \mathbf{f}_i' \left[\frac{M(\mathbf{f}_1\mathbf{f}_1' + \mathbf{f}_2\mathbf{f}_2')}{(\sigma^{(0)})_\omega^2} + \frac{N(\mathbf{f}_3\mathbf{f}_3' + \mathbf{f}_4\mathbf{f}_4')}{(\sigma^{(0)})_D^2}\right]^{-1} \mathbf{f}_j, \quad i,j = 1,2.
$$

Then

$$\mathbf{S}_{(M_F \Sigma_{\vartheta_0} M_F)^+} = \begin{pmatrix} \boxed{11}, & \boxed{12} \\ \boxed{21}, & \boxed{22} \end{pmatrix} = \begin{pmatrix} 3.97 \times 10^{17}, & 5.9 \times 10^{10} \\ 5.9 \times 10^{10}, & 4.98 \times 10^{5} \end{pmatrix},$$

where

$$\boxed{11} = \frac{2M}{(\sigma_\omega^{(0)})^4} - 2M\frac{c_{1,1} + c_{2,2}}{(\sigma_\omega^{(0)})^6} + M^2\frac{c_{1,1}^2 + 2c_{1,2}^2 + c_{2,2}^2}{(\sigma_\omega^{(0)})^8},$$

$$\boxed{12} = MN\frac{c_{1,3}^2 + c_{2,3}^2 + c_{1,4}^2 + c_{2,4}^2}{(\sigma_\omega^{(0)})^4(\sigma_D^{(0)})^4} = \boxed{2,1},$$

$$\boxed{22} = \frac{2N}{(\sigma_D^{(0)})^4} - 2N\frac{c_{3,3} + c_{4,4}}{(\sigma_D^{(0)})^6} + N^2\frac{c_{3,3}^2 + 2c_{3,4}^2 + c_{4,4}^2}{(\sigma_D^{(0)})^8}.$$

Further

$$\begin{pmatrix} \mathbf{k}'_{1,\cdot} \\ \mathbf{k}'_{1,\cdot} \end{pmatrix} = \begin{pmatrix} \boxed{11}, & \boxed{12} \\ \boxed{21}, & \boxed{22} \end{pmatrix}^{-1} \left[\mathrm{Diag}\left\{ \left(\frac{10}{206264.806}\right)^2, 0.05^2 \right\} \right]^{-1}$$
$$= \begin{pmatrix} 1.09 \times 10^{-9}, & -1.21 \times 10^{-10} \\ -1.29 \times 10^{-4}, & 8.18 \times 10^{-4} \end{pmatrix}.$$

Let $\varepsilon_2 = 0.05$. Then

$$\varepsilon_1 = \sqrt{0.05}\min\left\{ \sqrt{\frac{\sqrt{\left\{2\mathbf{S}^{-1}_{(M_F\Sigma_{\vartheta_0}M_F)^+}\right\}_{1,1}}}{\sum_{j=1}^p |k_{1,j}|}}, \sqrt{\frac{\sqrt{\left\{2\mathbf{S}^{-1}_{(M_F\Sigma_{\vartheta_0}M_F)^+}\right\}_{2,2}}}{\sum_{j=1}^p |k_{2,j}|}} \right\}$$
$$= 0.522576$$

and

$$\mathcal{L}_b = \left\{ \delta\boldsymbol{\beta} : \delta\boldsymbol{\beta}'\mathbf{F}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{F}\delta\boldsymbol{\beta} \le \frac{2\varepsilon_1}{K_{\vartheta_0}^{(par)}(\boldsymbol{\beta}_0)} \right\}$$
$$= \left\{ \delta\boldsymbol{\beta} : \delta\boldsymbol{\beta}'\mathbf{F}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{F}\delta\boldsymbol{\beta} \le 21280.6 \right\},$$

$$\mathcal{L}_\vartheta = \left\{ \delta\boldsymbol{\beta} : \delta\boldsymbol{\beta}'\mathbf{F}'\boldsymbol{\Sigma}_{\vartheta_0}^{-1}\mathbf{F}\delta\boldsymbol{\beta} \le \frac{2\varepsilon_1}{K_{\vartheta_0}^{(int)}(\boldsymbol{\beta}_0)} \right\}$$
$$= \left\{ \delta\boldsymbol{\beta} : \delta\boldsymbol{\beta}'\mathbf{F}'\boldsymbol{\Sigma}_{\vartheta_0}^{-1}\mathbf{F}\delta\boldsymbol{\beta} \le 32735.1 \right\}.$$

Now we can check whether $\mathcal{E}_\beta \subset \mathcal{L}_\vartheta$. At least it must be satisfied the inequality $\chi_k^2(0; 1-\alpha) = \chi_2^2(0; 0.95) = 5.99 \ll 2\varepsilon_1/K_{\vartheta_0}^{(int)}(\boldsymbol{\beta}_0) = 32735.1$.

Now it is necessary to check the inclusion $\mathcal{E}_\beta \subset \mathcal{L}_b \cap \mathcal{L}_\vartheta$. If $1 - \alpha = 0.95$, $\varepsilon_1 = 0.522576$, then the situation is given in Fig. 1.

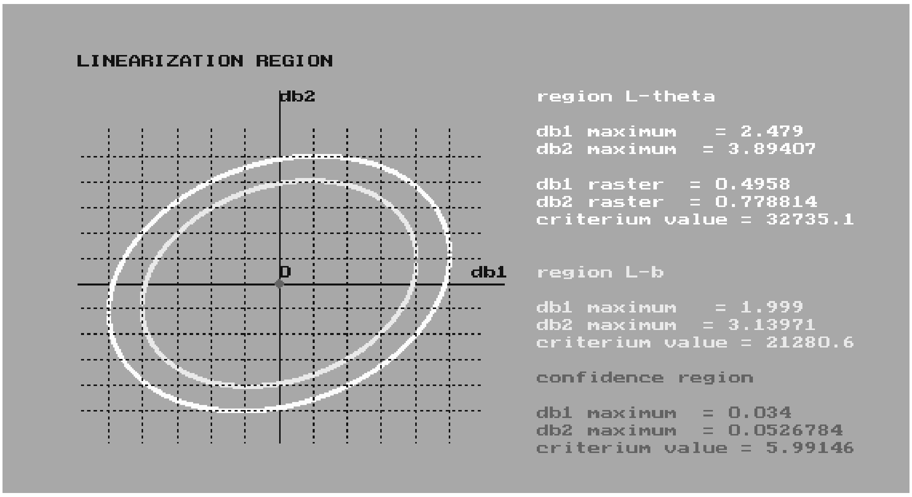

Fig. 1: Regions $\mathcal{L}_b$, $\mathcal{L}_\vartheta$ (with the same ε_1) and $\mathcal{E}_\beta$ (for $1-\alpha = 0.95$).

As far as the linearization is concerned, there is no problem, since the region $\mathcal{L}_b$ and $\mathcal{L}_\vartheta$ are very large in a comparison with the confidence ellipse $\mathcal{E}_\beta$.

Now it is to be checked the inclusions $\mathcal{K}_\vartheta \subset \mathcal{N}_{h_i}$, $i = 1, 2$. We need the matrices

$$\mathbf{W}_{h_i} = \begin{pmatrix} \mathbf{L}'_{h_i}\mathbf{V}_1 \\ \mathbf{L}'_{h_i}\mathbf{V}_2 \end{pmatrix} [\mathbf{M}_F(\boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0)\mathbf{M}_F]^+(\mathbf{V}_1\mathbf{L}_{h_i}, \mathbf{V}_p\mathbf{L}_{h_i}),$$

where

$$\mathbf{L}_{h_i} = \mathbf{h}'_i[\mathbf{F}'\boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0)^{-1}\mathbf{F}]^{-1}\mathbf{F}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0), \quad \mathbf{h}_1 = (1,0)', \quad \mathbf{h}_2 = (0,1).$$

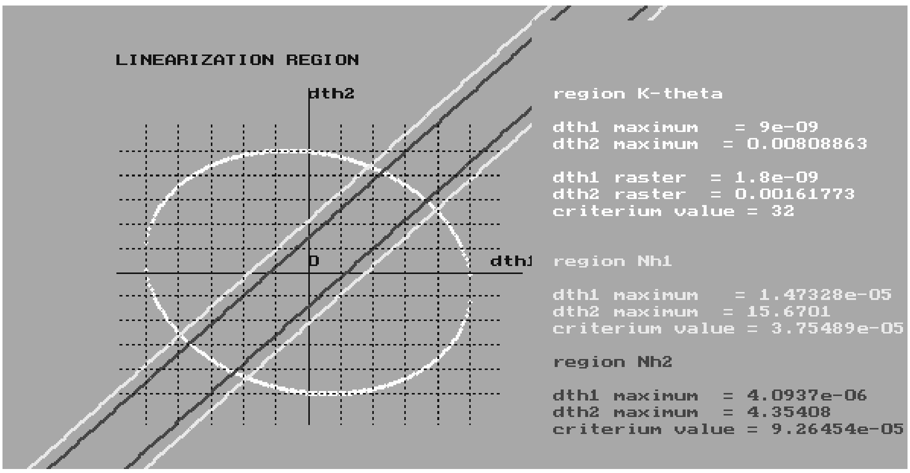

Fig. 2: Regions $\mathcal{K}_\vartheta$ for $t = 4$, $\mathcal{N}_{h_1}$ and $\mathcal{N}_{h_2}$ for $\varepsilon_3 = 0.1$.

Further

$$\begin{aligned}
\mathcal{K}_\vartheta &= \left\{\delta\boldsymbol{\vartheta} : \delta\boldsymbol{\vartheta}'\mathbf{S}_{(M_F\Sigma_{\vartheta_0}M_F)^+}\delta\boldsymbol{\vartheta} \le 2t^2\right\} \\
&= \left\{\delta\boldsymbol{\vartheta} : \delta\boldsymbol{\vartheta}'\mathbf{S}_{(M_F\Sigma_{\vartheta_0}M_F)^+}\delta\boldsymbol{\vartheta} \le 32\right\}, \\
\mathcal{N}_{h_1} &= \left\{\delta\boldsymbol{\vartheta} : \delta\boldsymbol{\vartheta}'\mathbf{W}_{h_1}\delta\boldsymbol{\vartheta} \le 2\varepsilon_3\mathbf{h}_1'(\mathbf{F}'\boldsymbol{\Sigma}_{\vartheta_0}^{-1}\mathbf{F})^{-1}\mathbf{h}_1'\right\} \\
&= \left\{\delta\boldsymbol{\vartheta} : \delta\boldsymbol{\vartheta}'\mathbf{W}_{h_1}\delta\boldsymbol{\vartheta} \le 3.75489 \times 10^{-5}\right\}, \\
\mathcal{N}_{h_2} &= \left\{\delta\boldsymbol{\vartheta} : \delta\boldsymbol{\vartheta}'\mathbf{W}_{h_2}\delta\boldsymbol{\vartheta} \le 2\varepsilon_3\mathbf{h}_2'(\mathbf{F}'\boldsymbol{\Sigma}_{\vartheta_0}^{-1}\mathbf{F})^{-1}\mathbf{h}_2'\right\} \\
&= \left\{\delta\boldsymbol{\vartheta} : \delta\boldsymbol{\vartheta}'\mathbf{W}_{h_2}\delta\boldsymbol{\vartheta} \le 9.26454 \times 10^{-5}\right\}.
\end{aligned}$$

For $\varepsilon_3 = 0.1$ and $t = 4$, see Fig. 2.

As far as the sensitiveness is concerned, the situation is more complicated. Fig. 2 shows that an accuracy of the estimators $\hat{\vartheta}_1$ and $\hat{\vartheta}_2$ based on the measurement results only is not sufficient. It is necessary to realize an additional experiment for the more accurate estimation of the parameters ϑ_1 and ϑ_2.

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Dually Residuated ℓ-monoids Having No Non-trivial Convex Subalgebras*

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Abstract

In this note we describe the structure of dually residuated ℓ-monoids ($DR\ell$-monoids) that have no non-trivial convex subalgebras.

Key words: $DR\ell$-monoid; $GPMV$-algebra; Archimedean property.

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A *dually residuated ℓ-monoid*, a *$DR\ell$-monoid* for short, is an algebra

$$(A, \oplus, 0, \vee, \wedge, \oslash, \otimes)$$

of type $\langle 2, 0, 2, 2, 2, 2\rangle$ such that

(a) $(A, \oplus, 0, \vee, \wedge)$ is a lattice-ordered monoid, i.e., $(A, \oplus, 0)$ is a monoid, $(A, \vee, \wedge)$ is a lattice and $\oplus$ distributes over both $\vee$ and $\wedge$,

(b) for any $a, b \in A$, $a \oslash b$ is the least element $x \in A$ with $x \oplus b \geq a$, and $a \otimes b$ is the least element $y \in A$ with $b \oplus y \geq a$, and

(c) A satisfies the identities

$$((x \oslash y) \vee 0) \oplus y \leq x \vee y, \quad y \oplus ((x \otimes y) \vee 0) \leq x \vee y,$$
$$x \oslash x \geq 0, \quad x \otimes x \geq 0.$$

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If the operation $\oplus$ is commutative then A is called a *commutative DRℓ-monoid*. In such a case, the operations $\oslash$ and $\varobslash$ coincide, and also conversely, A is commutative whenever $\oslash = \varobslash$.

Commutative *DRℓ*-monoids were originally introduced by K. L. N. Swamy [10] in order to capture the common features of Abelian ℓ-groups and Boolean algebras. The above definition, omitting the commutativity of $\oplus$, is due to T. Kovář [6] and allows us to consider all ℓ-groups in the setting of *DRℓ*-monoids. Indeed, given an arbitrary ℓ-group $(G, +, -, 0, \vee, \wedge)$, then $(G, +, 0, \vee, \wedge, \oslash, \varobslash)$ is a *DRℓ*-monoid in which $x \oslash y := x - y$ and $x \varobslash y := -y + x$.

The reader familiar with residuated lattices easily recognizes that the name "dually residuated ℓ-monoid" says less than the definition since *DRℓ*-monoids are equivalent to a certain proper subclass of residuated lattices. To be more precise, by a *residuated lattice* we mean an algebra $(L, \cdot, e, \vee, \wedge, \rightarrow, \rightsquigarrow)$ of type $\langle 2, 0, 2, 2, 2, 2\rangle$, where $(L, \cdot, e)$ is a monoid, $(L, \vee, \wedge)$ is a lattice and the equivalences

$$a \cdot b \leq c \quad \text{iff} \quad a \leq b \rightarrow c \quad \text{iff} \quad b \leq a \rightsquigarrow c \tag{1}$$

hold for all $a, b, c \in L$. Though it need not be evident at once, it not hard to show that our *DRℓ*-monoids are termwise equivalent to those residuated lattices satisfying the identities

$$x \wedge y = ((x \rightarrow y) \wedge e) \cdot x = x \cdot ((x \rightsquigarrow y) \wedge e). \tag{2}$$

Residuated lattices that fulfil (2) were considered e.g. in [2], [5] under the name *GBL-algebras.*

Now, we shortly review some relevant concepts from [7]. Given a *DRℓ*-monoid A, we define the *absolute value* of $x \in A$ by

$$|x| := x \vee (0 \oslash x) = x \vee (0 \varobslash x).$$

A non-empty subset I of A is called an *ideal* if

(I1) $a \oplus b \in I$ for all $a, b \in I$,

(I2) $a \in I$ and $|b| \leq |a|$ imply $b \in I$.

By a *non-trivial* ideal of we mean an ideal I with $\{0\} \subset I \subset A$.

The set $\mathscr{I}(A)$ of all ideals of A partially ordered by set-inclusion forms an algebraic distributive lattice in which infima agree with set-theoretical intersections. Hence for every $X \subseteq A$ there exists the smallest ideal $I(X)$ containing X; for $\emptyset \neq X$ we have

$$I(X) = \{a \in A : |a| \leq |x_1| \oplus \cdots \oplus |x_n| \text{ for some } x_1, \ldots, x_n \in X, n \in \mathbb{N}\}.$$

It can be easily proved that $I \subseteq A$ is an ideal if and only if I is a convex subalgebra of A.

The congruence kernels are characterized as the so-called *normal ideals*: An ideal I of A is said to be *normal* if

$$a \oslash b \in I \quad \text{iff} \quad a \varobslash b \in I$$

for every $a, b \in A$. If I is a normal ideal then the relation Θ_I defined via

$$(a, b) \in \Theta_I \quad \text{iff} \quad (a \oslash b) \vee (b \oslash a) \in I$$

is a congruence with $[0]_{\Theta_I} = I$, and conversely, for any congruence Θ on A, $I = [0]_\Theta$ is a normal ideal such that $\Theta_I = \Theta$. Therefore, the congruence lattice of A is isomorphic to the lattice of all normal ideals of A. For the sake of brevity, we write A/I for the quotient algebra A/Θ, where $I = [0]_\Theta$, and the elements of A/I are denoted by a/I rather than $[a]_\Theta$.

There are two basic kinds of $DR\ell$-monoids from which every $DR\ell$-monoid can be built using direct products: ℓ-groups and lower bounded $DR\ell$-monoids, i.e., $DR\ell$-monoids having 0 as a least element.

Let A be an arbitrary $DR\ell$-monoid. Put

$$G_A := \{a \in A : a \oplus (0 \oslash a) = 0 = (0 \oslash a) \oplus a\}$$

and

$$S_A := \{a \in A : 0 \oslash a = 0\} = \{a \in A : 0 \mathbin{\overline{\oslash}} a = 0\}.$$

Both G_A and S_A are ideals of A; obviously, the first one is an ℓ-group and the second one is a lower bounded $DR\ell$-monoid. T. Kovář proved in [6] that A is the direct sum of G_A and S_A. The same result for GBL-algebras was independently obtained by N. Galatos and C. Tsinakis (see [2]).

Assume that a $DR\ell$-monoid A has no non-trivial ideals. Since both G_A and S_A are (normal) ideals of A, it is clear that either $A = G_A$ or $A = S_A$. In the former case, A is an ℓ-group having no non-trivial convex ℓ-subgroups, and hence it is an Archimedean totally ordered group which is isomorphic to a subgroup of the additive group of reals equipped with the usual order. Therefore, in the sequel we concentrate on lower bounded $DR\ell$-monoids which have no non-trivial ideals.

For every $x, y \in A$ and $n \in \mathbb{N}_0$, we inductively define

$$0 \odot x := 0, \qquad (n+1) \odot x := n \odot x \oplus x,$$

and

$$x \oslash^0 y := x, \qquad x \oslash^{n+1} y := (x \oslash^n y) \oslash y;$$

$x \mathbin{\overline{\oslash}}^n y$ is defined analogously.

Lemma 1 *Let A be a lower bounded $DR\ell$-monoid. The following are equivalent:*

(a) *A has no non-trivial ideals;*

(b) *for every $a, b \in A$, $a \neq 0$, there exists $n \in \mathbb{N}$ such that $b \leq n \odot a$;*

(c) *for every $a, b \in A$, $a \neq 0$, there exists $n \in \mathbb{N}$ such that $b \oslash^n a = 0$;*

(d) *for every $a, b \in A$, $a \neq 0$, there exists $n \in \mathbb{N}$ such that $b \mathbin{\overline{\oslash}}^n a = 0$.*

Proof Obviously, (b)–(d) are equivalent. Moreover, since

$$I(a) := I(\{a\}) = \{b \in A : b \leq n \odot a \text{ for some } n \in \mathbb{N}\},$$

it follows that each of these conditons is equivalent to (a). □

Lemma 2 *Let A be a lower bounded DRℓ-monoid and H be its normal ideal. Then the ideal lattice $\mathscr{I}(A/H)$ of the quotient DRℓ-monoid A/H is isomorphic to the interval $[H, A]$ of the lattice $\mathscr{I}(A)$.*

Proof If $I \in \mathscr{I}(A)$ and $H \subseteq I$ then

$$\phi(I) := \{x/H : x \in I\}$$

is an ideal of A/H. Conversely, if $J \in \mathscr{I}(A/H)$ then

$$\psi(J) := \{x \in A : x/H \in J\}$$

is an ideal of A such that $H \subseteq \psi(J)$. It is easily seen that the mappings ϕ and ψ are mutually inverse order-preserving bijections between $\mathscr{I}(A/H)$ and $[H, A]$ ordered by set-theoretical inclusion. □

An ideal $I \in \mathscr{I}(A)$ is called *maximal* if $I \subset A$ and there is no ideal $J \in \mathscr{I}(A)$ such that $I \subset J \subset A$. In view of Lemma 2 we have:

Proposition 3 *Let A be a lower bounded DRℓ-monoid and H be a normal ideal with $H \subset A$. Then H is maximal if and only if the quotient DRℓ-monoid A/H has no non-trivial ideals.*

Lemma 4 *Let A be a lower bounded DRℓ-monoid that has no non-trivial ideals. Then for every $a, b \in A$, $a \neq 0$,*

$$a \oslash b = a \Longrightarrow b = 0, \qquad a \otimes b = a \Longrightarrow b = 0.$$

Proof We show that the set

$$J_a := \{x \in A : a \oslash x = a\}$$

is an ideal of A. Clearly, $0 \in J_a$. If $x, y \in J_a$ then $a \oslash (x \oplus y) = (a \oslash y) \oslash x = a \oslash x = a$, so that $x \oplus y \in J_a$. Finally, if $x \in J_a$ and $y \leq x$ then $a = a \oslash x \leq a \oslash y \leq a$, and hence $a = a \oslash y$.

However, since $a \notin J_a$ and A has no non-trivial ideals, it follows that $J_a = \{0\}$, and consequently, $a \oslash b = a$ entails $b = 0$ as claimed. □

Lemma 5 *Let A be a lower bounded DRℓ-monoid having no non-trivial ideals. If $0 < x \leq y < a$ and $a \oslash x = a \oslash y$ or $a \otimes x = a \otimes y$, then $x = y$.*

Proof We have $y = x \vee y = (y \oslash x) \oplus x$, so that $a \oslash x = a \oslash y = a \oslash ((y \oslash x) \oplus x) = (a \oslash x) \oslash (y \oslash x)$. Since $a \oslash x \neq 0$, we obtain $y \oslash x = 0$ by Lemma 4, yielding $y \leq x$, so $x = y$. □

Theorem 6 *Let A be a DRℓ-monoid that has no non-trivial ideals. Then A satisfies the identities*

$$x \wedge y = x \oslash ((x \odot y) \vee 0) = x \odot ((x \oslash y) \vee 0). \tag{3}$$

Proof In the case when A is an ℓ-group the identities (3) evidently hold. Hence assume that A is a lower bounded *DRℓ*-monoid. Note that $x \oslash ((x \odot y) \vee 0) = x \oslash (x \odot y)$ and $x \odot ((x \oslash y) \vee 0) = x \odot (x \oslash y)$. If $x \leq y$ then $x \oslash (x \odot y) = x \oslash 0 = x = x \wedge y$ and also $x \odot (x \oslash y) = x = x \wedge y$. Further, let $x \nleq y$, i.e., $x \wedge y < x$. Since both $x \oslash (x \odot y)$ and $x \odot (x \oslash y)$ are common lower bounds of $\{x, y\}$, we may suppose that $0 < x \wedge y < x$. In this case we have $0 < x \oslash (x \odot y) \leq x \wedge y < x$ because $x \oslash (x \odot y) = 0$ would mean $x = x \oslash y$ yielding $y = 0$ which is impossible due to $0 < x \wedge y$. Finally, we have $x \odot (x \oslash (x \odot y)) = x \odot y = x \odot (x \wedge y)$ which entails $x \oslash (x \odot y) = x \wedge y$ by Lemma 5. By replacing $\oslash$ and $\odot$ we get $x \odot (x \oslash y) = x \wedge y$. □

Therefore, a *DRℓ*-monoid without non-trivial ideals is either an ℓ-group or is lower bounded and verifies the identities

$$x \wedge y = x \oslash (x \odot y) = x \odot (x \oslash y). \tag{4}$$

Such *DRℓ*-monoids were investigated in [8], [9] and called here *generalized pseudo MV-algebras* (*GPMV-algebras* for short). The name is motived by the fact that bounded *GPMV*-algebras are termwise equivalent to pseudo *MV*-algebras. In the literature, there exist two classes of algebras that are equivalent to *GPMV*-algebras, namely, *integral GMV-algebras* and *Wajsberg pseudo-hoops* (see [2] and [3], respectively).

By [9], every *GPMV*-algebra A can be embedded into the positive cone $G(A)^+$ of an ℓ-group $G(A)$ such that, assuming $A \subseteq G(A)$, A is a lattice ideal of $G(A)^+$ which generates $G(A)^+$ as a semigroup, and the operations $\oslash$, $\odot$ on A are given as follows:

$$a \oslash b := (a - b) \vee 0, \qquad a \odot b := (-b + a) \vee 0.$$

Moreover, the ideal lattice $\mathscr{I}(A)$ of A and the lattice $\mathscr{C}(G(A))$ of all convex ℓ-subgroups of $G(A)$ are isomorphic under the mapping assigning to each $I \in \mathscr{I}(A)$ the convex ℓ-subgroup of $G(A)$ generated by I. In view of the well-known fact that an ℓ-group is totally ordered exactly if its lattice of all convex ℓ-subgroups is a chain, this means that A is totally ordered if and only if so is $G(A)$, and hence we gain:

Corollary 7 *Every DRℓ-monoid which has no non-trivial ideals is totally ordered.*

In [9], the Archimedean property for *GPMV*-algebra is defined in the following way. Given a *GPMV*-algebra A, we introduce a partial addition $+$ by setting $a + b := a \oplus b$ iff $(a \oplus b) \oslash b = a$, or equivalently, $(a \oplus b) \odot a = b$. Observe that if $A \subseteq G(A)$, then $+$ is the restriction of the group addition to those pairs of elements of A whose sum belongs to A.

This partial operation is associative in the sense that $a + b$ and $(a + b) + c$ exist iff $b + c$ and $a + (b + c)$ exist and $(a + b) + c = a + (b + c)$, and therefore, for any $a \in A$, $n \in \mathbb{N}_0$, we may define

$$0 \cdot a := 0, \qquad (n + 1) \cdot a := n \cdot a + a.$$

Accordingly, we write $a \ll b$ whenever $n \cdot a$ exists and $n \cdot a \leq b$ for all $n \in \mathbb{N}$. Now, we say that a *GPMV*-algebra A is *Archimedean* if $a \not\ll b$ for all $a, b \in A \setminus \{0\}$.

As proved in [9], a *GPMV*-algebra A is Archimedean if and only if $G(A)$ is an Archimedean ℓ-group, hence all Archimedean *GPMV*-algebras are commutative. Therefore we conclude:

Theorem 8 *Let A be a DRℓ-monoid having no non-trivial ideals. Then A is either an Archimedean totally ordered group or A is Archimedean totally ordered GPMV-algebra.*

In fact, if A is a totally ordered Archimedean *GPMV*-algebra then the ℓ-group $G(A)$ is isomorphic to a subgroup of the additive group $\mathbb{R}$ of real numbers with the usual order, and consequently, we may always assume that A is a subset of $\mathbb{R}^+$; the operations $\oslash$ and $\circledcirc$ agree and we have $a \oslash b = a \circledcirc b = \max\{a - b, 0\}$.

Corollary 9 *Let A be a lower bounded DRℓ-monoid. If H is a normal ideal of A which is simultaneously a maximal ideal, then A/H is a totally ordered Archimedean GPMV-algebra.*

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Linear Model with Nuisance Parameters and with Constraints on Useful and Nuisance Parameters*

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Abstract

The properties of the regular linear model are well known (see [1], Chapter 1). In this paper the situation where the vector of the first order parameters is divided into two parts (to the vector of the useful parameters and to the vector of the nuisance parameters) is considered. It will be shown how the BLUEs of these parameters will be changed by constraints given on them. The theory will be illustrated by an example from the practice.

Key words: Regular linear regression model; nuisance parameters; BLUE; constraints.

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1 Introduction, notations

The following notation will be used throughout the paper:

R^n	the space of all n-dimensional real vectors;
$\boldsymbol{u}_p$, $\boldsymbol{A}_{m,n}$	the real column p-dimensional vector, the real $m \times n$ matrix;

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$\boldsymbol{A}', r(\boldsymbol{A})$	the transpose, the rank of the matrix $\boldsymbol{A}$;
$\boldsymbol{A}_{[r,s]}$	rs-th element of matrix $\boldsymbol{A}$;
$\mathscr{M}(\boldsymbol{A}), \mathrm{Ker}(\boldsymbol{A})$	the column space, the null space of the matrix $\boldsymbol{A}$;
$\boldsymbol{A}^-$	a generalized inverse of a matrix $\boldsymbol{A}$ (satisfying $\boldsymbol{A}\boldsymbol{A}^-\boldsymbol{A} = \boldsymbol{A}$);
$\boldsymbol{A}^+$	the Moore–Penrose generalized inverse of a matrix $\boldsymbol{A}$ (satisfying $\boldsymbol{A}\boldsymbol{A}^+\boldsymbol{A} = \boldsymbol{A}$, $\boldsymbol{A}^+\boldsymbol{A}\boldsymbol{A}^+ = \boldsymbol{A}^+$, $(\boldsymbol{A}\boldsymbol{A}^+)' = \boldsymbol{A}\boldsymbol{A}^+$, $(\boldsymbol{A}^+\boldsymbol{A})' = \boldsymbol{A}^+\boldsymbol{A}$);
$\boldsymbol{P}_A$	the orthogonal projector in the Euclidean norm onto $\mathscr{M}(\boldsymbol{A})$;
$\boldsymbol{M}_A = \boldsymbol{I} - \boldsymbol{P}_A$	the orthogonal projector in the Euclidean norm onto $\mathscr{M}^{\perp}(\boldsymbol{A}) = \mathrm{Ker}(\boldsymbol{A}')$;
$\boldsymbol{I}_k$	the $k \times k$ identity matrix;
$0_{m,n}$	the $m \times n$ null matrix;
$\boldsymbol{o}$	the null vector;
$1_k = (1, \ldots, 1)' \in R^k$.	

If $\mathscr{M}(\boldsymbol{A}) \subset \mathscr{M}(\boldsymbol{S})$, $\boldsymbol{S}$ p.s.d., then the symbol $\boldsymbol{P}_A^{S^-}$ denotes the projector projecting vectors in $\mathscr{M}(\boldsymbol{S})$ onto $\mathscr{M}(\boldsymbol{A})$ along $\mathscr{M}(\boldsymbol{S}\boldsymbol{A}^{\perp})$. A general representation of all such projectors $\boldsymbol{P}_A^{S^-}$ is given by $\boldsymbol{A}(\boldsymbol{A}'\boldsymbol{S}^-\boldsymbol{A})^-\boldsymbol{A}'\boldsymbol{S}^- + \boldsymbol{B}(\boldsymbol{I} - \boldsymbol{S}\boldsymbol{S}^-)$, where $\boldsymbol{B}$ is arbitrary, (see [3], (2.14)). $\boldsymbol{M}_A^{S^-} = \boldsymbol{I} - \boldsymbol{P}_A^{S^-}$.

Assertion 1 (see [1], Lemma 10.1.35) *Let $\boldsymbol{X}$ be any $n \times k$ matrix and Σ an $n \times n$ p.s.d. matrix.*
(i) If Σ is p.d., then

$$(\boldsymbol{M}_X \Sigma \boldsymbol{M}_X)^+ = \Sigma^{-1} - \Sigma^{-1}\boldsymbol{X}(\boldsymbol{X}'\Sigma^{-1}\boldsymbol{X})^-\boldsymbol{X}'\Sigma^{-1} = \Sigma^{-1}\boldsymbol{M}_X^{\Sigma^{-1}}.$$

$$(ii) \qquad \begin{aligned}(\boldsymbol{M}_X \Sigma \boldsymbol{M}_X)^+ &= \boldsymbol{M}_X(\boldsymbol{M}_X \Sigma \boldsymbol{M}_X)^+ = (\boldsymbol{M}_X \Sigma \boldsymbol{M}_X)^+\boldsymbol{M}_X \\ &= \boldsymbol{M}_X(\boldsymbol{M}_X \Sigma \boldsymbol{M}_X)^+\boldsymbol{M}_X.\end{aligned}$$

2 Best linear unbiased estimators

Let us consider the following linear model

$$\boldsymbol{Y} = (\boldsymbol{X}, \boldsymbol{S})\begin{pmatrix}\beta \\ \kappa\end{pmatrix} + \varepsilon, \tag{1}$$

where $\boldsymbol{Y} = (\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_n)'$ is a random observation vector; $\beta \in R^k$ is a vector of the useful parameters; $\kappa \in R^l$ is a vector of the nuisance parameters; $\boldsymbol{X}_{n,k}$ is a design matrix belonging to the vector β; $\boldsymbol{S}_{n,l}$ is a design matrix belonging to the vector κ.

We suppose that
1. $E(\boldsymbol{Y}) = \boldsymbol{X}\beta + \boldsymbol{S}\kappa$, $\forall \beta \in R^k$, $\forall \kappa \in R^l$,
2. $\mathrm{var}(\boldsymbol{Y}) = \Sigma$ is a known matrix,
3. matrix Σ is not a function of the vector $(\beta', \kappa')'$.

If matrix Σ is positive definite and $r(\boldsymbol{X}, \boldsymbol{S}) = k + l < n$, the model is said to be *regular*, (see [1], p. 13).

Theorem 1 *In the regular model (1) the BLUEs of the parameters are given as*

$$\begin{aligned}\hat{\beta} &= \boldsymbol{C}^{-1}\boldsymbol{X}'\Sigma^{-1}\boldsymbol{Y}\\ &\quad - \boldsymbol{C}^{-1}\boldsymbol{X}'\Sigma^{-1}\boldsymbol{S}[\boldsymbol{S}'(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^{+}\boldsymbol{S}]^{-1}\boldsymbol{S}'\Sigma^{-1}\{\boldsymbol{I} - \boldsymbol{X}\boldsymbol{C}^{-1}\boldsymbol{X}'\Sigma^{-1}\}\boldsymbol{Y}\\ &= \boldsymbol{C}^{-1}\boldsymbol{X}'\Sigma^{-1}\{\boldsymbol{I} - \boldsymbol{S}[\boldsymbol{S}'(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^{+}\boldsymbol{S}]^{-1}\boldsymbol{S}'(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^{+}\}\boldsymbol{Y}, \qquad (2)\end{aligned}$$

$$\begin{aligned}\hat{\kappa} &= [\boldsymbol{S}'(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^{+}\boldsymbol{S}]^{-1}\boldsymbol{S}'\Sigma^{-1}\{\boldsymbol{I} - \boldsymbol{X}\boldsymbol{C}^{-1}\boldsymbol{X}'\Sigma^{-1}\}\boldsymbol{Y}\\ &= [\boldsymbol{S}'(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^{+}\boldsymbol{S}]^{-1}\boldsymbol{S}'(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^{+}\boldsymbol{Y}, \qquad (3)\end{aligned}$$

where $\boldsymbol{C} = \boldsymbol{X}'\Sigma^{-1}\boldsymbol{X}$.

Proof According to the Theorem 1.1.1 in [1] and using the following Rohde's formula for inverse of partitioned p.d. matrix (see [1], Lemma 10.1.40)

$$\begin{pmatrix}\boldsymbol{F}, & \boldsymbol{G}\\ \boldsymbol{G}', & \boldsymbol{H}\end{pmatrix}^{-1}$$

$$= \begin{pmatrix}\boldsymbol{F}^{-1} + \boldsymbol{F}^{-1}\boldsymbol{G}(\boldsymbol{H} - \boldsymbol{G}'\boldsymbol{F}^{-1}\boldsymbol{G})^{-1}\boldsymbol{G}'\boldsymbol{F}^{-1}, & -\boldsymbol{F}^{-1}\boldsymbol{G}(\boldsymbol{H} - \boldsymbol{G}'\boldsymbol{F}^{-1}\boldsymbol{G})^{-1}\\ -(\boldsymbol{H} - \boldsymbol{G}'\boldsymbol{F}^{-1}\boldsymbol{G})^{-1}\boldsymbol{G}'\boldsymbol{F}^{-1}, & (\boldsymbol{H} - \boldsymbol{G}'\boldsymbol{F}^{-1}\boldsymbol{G})^{-1}\end{pmatrix} \qquad (4)$$

the BLUE of the vector parameter $(\beta', \kappa')'$ is given by

$$\begin{pmatrix}\hat{\beta}\\ \hat{\kappa}\end{pmatrix} = \left[\begin{pmatrix}\boldsymbol{X}'\\ \boldsymbol{S}'\end{pmatrix}\Sigma^{-1}(\boldsymbol{X}, \boldsymbol{S})\right]^{-1}\begin{pmatrix}\boldsymbol{X}'\\ \boldsymbol{S}'\end{pmatrix}\Sigma^{-1}\boldsymbol{Y}$$

$$= \begin{bmatrix}\boldsymbol{X}'\Sigma^{-1}\boldsymbol{X}, & \boldsymbol{X}'\Sigma^{-1}\boldsymbol{S}\\ \boldsymbol{S}'\Sigma^{-1}\boldsymbol{X}, & \boldsymbol{S}'\Sigma^{-1}\boldsymbol{S}\end{bmatrix}^{-1}\begin{pmatrix}\boldsymbol{X}'\Sigma^{-1}\\ \boldsymbol{S}'\Sigma^{-1}\end{pmatrix}\boldsymbol{Y} = \begin{pmatrix}\boxed{11}, & \boxed{12}\\ \boxed{21}, & \boxed{22}\end{pmatrix}\begin{pmatrix}\boldsymbol{X}'\Sigma^{-1}\boldsymbol{Y}\\ \boldsymbol{S}'\Sigma^{-1}\boldsymbol{Y}\end{pmatrix},$$

where

$\boxed{11} = \boldsymbol{C}^{-1} + \boldsymbol{C}^{-1}\boldsymbol{X}'\Sigma^{-1}\boldsymbol{S}[\boldsymbol{S}'(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^{+}\boldsymbol{S}]^{-1}\boldsymbol{S}'\Sigma^{-1}\boldsymbol{X}\boldsymbol{C}^{-1}$,

$\boxed{12} = -\boldsymbol{C}^{-1}\boldsymbol{X}'\Sigma^{-1}\boldsymbol{S}[\boldsymbol{S}'(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^{+}\boldsymbol{S}]^{-1}$,

$\boxed{21} = -[\boldsymbol{S}'(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^{+}\boldsymbol{S}]^{-1}\boldsymbol{S}'\Sigma^{-1}\boldsymbol{X}\boldsymbol{C}^{-1}$,

$\boxed{22} = [\boldsymbol{S}'(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^{+}\boldsymbol{S}]^{-1}$.

As Σ is supposed to be positive definite, we utilized Assertion 1, (i). The rest of the proof is obvious. □

Theorem 2 *For the estimators* $\hat{\beta}$, $\hat{\kappa}$ *is valid*

$$\begin{aligned}\text{var}(\hat{\beta}) &= \boldsymbol{C}^{-1} + \boldsymbol{C}^{-1}\boldsymbol{X}'\Sigma^{-1}\boldsymbol{S}[\boldsymbol{S}'(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^{+}\boldsymbol{S}]^{-1}\boldsymbol{S}'\Sigma^{-1}\boldsymbol{X}\boldsymbol{C}^{-1}, \qquad (5)\\ \text{var}(\hat{\kappa}) &= [\boldsymbol{S}'(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^{+}\boldsymbol{S}]^{-1}, \qquad (6)\\ \text{cov}(\hat{\beta}, \hat{\kappa}) &= -\boldsymbol{C}^{-1}\boldsymbol{X}'\Sigma^{-1}\boldsymbol{S}[\boldsymbol{S}'(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^{+}\boldsymbol{S}]^{-1}. \qquad (7)\end{aligned}$$

Proof

$$\begin{aligned}
\operatorname{var}(\hat{\beta}) &= \boldsymbol{C}^{-1}\boldsymbol{X}'\Sigma^{-1}\{\boldsymbol{I}-\boldsymbol{S}[\boldsymbol{S}'(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^{+}\boldsymbol{S}]^{-1}\boldsymbol{S}'(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^{+}\}\Sigma \\
&\quad \times\{\boldsymbol{I}-(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^{+}\boldsymbol{S}[\boldsymbol{S}'(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^{+}\boldsymbol{S}]^{-1}\boldsymbol{S}'\}\Sigma^{-1}\boldsymbol{X}\boldsymbol{C}^{-1} \\
&= \boldsymbol{C}^{-1}+\boldsymbol{C}^{-1}\boldsymbol{X}'\Sigma^{-1}\boldsymbol{S}[\boldsymbol{S}'(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^{+}\boldsymbol{S}]^{-1}\boldsymbol{S}'\Sigma^{-1}\boldsymbol{X}\boldsymbol{C}^{-1}, \\
\operatorname{var}(\hat{\kappa}) &= [\boldsymbol{S}'(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^{+}\boldsymbol{S}]^{-1}\boldsymbol{S}'(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^{+}\boldsymbol{M}_X\Sigma\boldsymbol{M}_X(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^{+} \\
&\quad \times\boldsymbol{S}[\boldsymbol{S}'(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^{+}\boldsymbol{S}]^{-1}=[\boldsymbol{S}'(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^{+}\boldsymbol{S}]^{-1}, \\
\operatorname{cov}(\hat{\beta},\hat{\kappa}) &= \boldsymbol{C}^{-1}\boldsymbol{X}'\Sigma^{-1}\{\boldsymbol{I}-\boldsymbol{S}[\boldsymbol{S}'(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^{+}\boldsymbol{S}]^{-1}\boldsymbol{S}'(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^{+}\} \\
&\quad \times\Sigma(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^{+}\boldsymbol{S}[\boldsymbol{S}'(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^{+}\boldsymbol{S}]^{-1} \\
&= -\boldsymbol{C}^{-1}\boldsymbol{X}'\Sigma^{-1}\boldsymbol{S}[\boldsymbol{S}'(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^{+}\boldsymbol{S}]^{-1}.
\end{aligned}$$

In the course of the proof the Assertion 1, (ii) was used. □

Let us consider model (1) with constrains given on both parameters, i.e. the model

$$\boldsymbol{Y}=(\boldsymbol{X},\boldsymbol{S})\begin{pmatrix}\beta\\ \kappa\end{pmatrix}+\varepsilon,\quad \boldsymbol{b}+\boldsymbol{B}_1\beta+\boldsymbol{B}_2\kappa=\boldsymbol{o}, \tag{8}$$

where we suppose for the $q\times k$ matrix $\boldsymbol{B}_1$ and $q\times l$ matrix $\boldsymbol{B}_2$ that

$$r(\boldsymbol{B}_2)=l<q,\quad r(\boldsymbol{B}_1,\boldsymbol{B}_2)=q<k+l.$$

Theorem 3 *The BLUEs $\hat{\hat{\beta}}$, $\hat{\hat{\kappa}}$ of the parameters β,κ under the model (8) are given by*

$$\begin{aligned}
\hat{\hat{\beta}} &= \hat{\beta}-(\boldsymbol{C}^{-1}\boldsymbol{B}_1'+\boldsymbol{C}^{-1}\boldsymbol{X}'\Sigma^{-1}\boldsymbol{S}\boldsymbol{Z}^{-1}\boldsymbol{U}') \\
&\quad \times[\boldsymbol{B}_1\boldsymbol{C}^{-1}\boldsymbol{B}_1'+\boldsymbol{U}\boldsymbol{Z}^{-1}\boldsymbol{U}']^{-1}(\boldsymbol{B}_1\hat{\beta}+\boldsymbol{B}_2\hat{\kappa}+\boldsymbol{b}),
\end{aligned} \tag{9}$$

$$\hat{\hat{\kappa}} = \hat{\kappa}+\boldsymbol{Z}^{-1}\boldsymbol{U}'\left[\boldsymbol{B}_1\boldsymbol{C}^{-1}\boldsymbol{B}_1'+\boldsymbol{U}\boldsymbol{Z}^{-1}\boldsymbol{U}'\right]^{-1}(\boldsymbol{B}_1\hat{\beta}+\boldsymbol{B}_2\hat{\kappa}+\boldsymbol{b}), \tag{10}$$

where $\boldsymbol{U}=\boldsymbol{B}_1\boldsymbol{C}^{-1}\boldsymbol{X}'\Sigma^{-1}\boldsymbol{S}-\boldsymbol{B}_2$, $\boldsymbol{Z}=\boldsymbol{S}'(\boldsymbol{M}_X\Sigma\boldsymbol{M}_X)^{+}\boldsymbol{S}$ and where $\hat{\beta},\hat{\kappa}$ are given in Theorem 1.

Proof In the following regular model with constraints

$$\begin{gathered}
\boldsymbol{Y}\sim_n(\boldsymbol{A}\theta,\Sigma),\quad \boldsymbol{b}+\boldsymbol{B}\theta=\boldsymbol{o},\\
r(\boldsymbol{A}_{n,k})=k<n,\quad r(\boldsymbol{B}_{q,k})=q<k,\quad \Sigma \text{ p.d.},
\end{gathered}$$

there is (according [2], theorem 4.3.1) for the BLUE of the parameter θ

$$\begin{gathered}
\hat{\hat{\theta}}=\{\boldsymbol{I}-(\boldsymbol{A}'\Sigma^{-1}\boldsymbol{A})^{-1}\boldsymbol{B}'[\boldsymbol{B}(\boldsymbol{A}'\Sigma^{-1}\boldsymbol{A})^{-1}\boldsymbol{B}']^{-1}\boldsymbol{B}\}\hat{\theta}\\
-(\boldsymbol{A}'\Sigma^{-1}\boldsymbol{A})^{-1}\boldsymbol{B}'[\boldsymbol{B}(\boldsymbol{A}'\Sigma^{-1}\boldsymbol{A})^{-1}\boldsymbol{B}']^{-1}\boldsymbol{b},
\end{gathered}$$

where $\hat{\theta}=(\boldsymbol{A}'\Sigma^{-1}\boldsymbol{A})^{-1}\boldsymbol{A}'\Sigma^{-1}\boldsymbol{Y}$, is the BLUE of θ without constraints.

In the model (8) we have

$$\boldsymbol{A} \to (\boldsymbol{X}, \boldsymbol{S}), \quad \theta \to \begin{pmatrix} \beta \\ \kappa \end{pmatrix}, \quad \boldsymbol{B} \to (\boldsymbol{B}_1, \boldsymbol{B}_2).$$

Thus analogously

$$\begin{pmatrix} \hat{\hat{\beta}} \\ \hat{\hat{\kappa}} \end{pmatrix} = \left\{ \begin{pmatrix} \boldsymbol{I}, & \boldsymbol{0} \\ \boldsymbol{0}, & \boldsymbol{I} \end{pmatrix} - \left[\begin{pmatrix} \boldsymbol{X}' \\ \boldsymbol{S}' \end{pmatrix} \Sigma^{-1} (\boldsymbol{X}, \boldsymbol{S}) \right]^{-1} \begin{pmatrix} \boldsymbol{B}_1' \\ \boldsymbol{B}_2' \end{pmatrix} \right.$$

$$\left. \times \left[(\boldsymbol{B}_1, \boldsymbol{B}_2) \left\{ \begin{pmatrix} \boldsymbol{X}' \\ \boldsymbol{S}' \end{pmatrix} \Sigma^{-1} (\boldsymbol{X}, \boldsymbol{S}) \right\}^{-1} \begin{pmatrix} \boldsymbol{B}_1' \\ \boldsymbol{B}_2' \end{pmatrix} \right]^{-1} (\boldsymbol{B}_1, \boldsymbol{B}_2) \right\} \begin{pmatrix} \hat{\beta} \\ \hat{\kappa} \end{pmatrix}$$

$$- \left[\begin{pmatrix} \boldsymbol{X}' \\ \boldsymbol{S}' \end{pmatrix} \Sigma^{-1} (\boldsymbol{X}, \boldsymbol{S}) \right]^{-1} \begin{pmatrix} \boldsymbol{B}_1' \\ \boldsymbol{B}_2' \end{pmatrix} \left[(\boldsymbol{B}_1, \boldsymbol{B}_2) \left\{ \begin{pmatrix} \boldsymbol{X}' \\ \boldsymbol{S}' \end{pmatrix} \Sigma^{-1} (\boldsymbol{X}, \boldsymbol{S}) \right\}^{-1} \begin{pmatrix} \boldsymbol{B}_1' \\ \boldsymbol{B}_2' \end{pmatrix} \right]^{-1} \boldsymbol{b},$$

where $\hat{\beta}, \hat{\kappa}$ are given in Theorem 1.

Let us calculate first

$$\left[(\boldsymbol{B}_1, \boldsymbol{B}_2) \left\{ \begin{pmatrix} \boldsymbol{X}' \\ \boldsymbol{S}' \end{pmatrix} \Sigma^{-1} (\boldsymbol{X}, \boldsymbol{S}) \right\}^{-1} \begin{pmatrix} \boldsymbol{B}_1' \\ \boldsymbol{B}_2' \end{pmatrix} \right]^{-1}$$

$$= \left[(\boldsymbol{B}_1, \boldsymbol{B}_2) \begin{pmatrix} \boxed{11} & \boxed{12} \\ \boxed{21} & \boxed{22} \end{pmatrix} \begin{pmatrix} \boldsymbol{B}_1' \\ \boldsymbol{B}_2' \end{pmatrix} \right]^{-1}$$

$$= \left(\boldsymbol{B}_1 \boldsymbol{C}^{-1} \boldsymbol{B}_1' + \boldsymbol{U} [\boldsymbol{S}' (\boldsymbol{M}_X \Sigma \boldsymbol{M}_X)^+ \boldsymbol{S}]^{-1} \boldsymbol{U}' \right)^{-1},$$

where $\boldsymbol{U} = \boldsymbol{B}_1 \boldsymbol{C}^{-1} \boldsymbol{X}' \Sigma^{-1} \boldsymbol{S} - \boldsymbol{B}_2$ and where $\boxed{11}$, $\boxed{12}$, $\boxed{21}$, $\boxed{22}$ are given in the proof of Theorem 1. Further

$$\left[\begin{pmatrix} \boldsymbol{X}' \\ \boldsymbol{S}' \end{pmatrix} \Sigma^{-1} (\boldsymbol{X}, \boldsymbol{S}) \right]^{-1} \begin{pmatrix} \boldsymbol{B}_1' \\ \boldsymbol{B}_2' \end{pmatrix} = \begin{pmatrix} \boxed{11} & \boxed{12} \\ \boxed{21} & \boxed{22} \end{pmatrix} \begin{pmatrix} \boldsymbol{B}_1' \\ \boldsymbol{B}_2' \end{pmatrix}$$

$$= \begin{pmatrix} \boldsymbol{C}^{-1} \boldsymbol{B}_1' + \boldsymbol{C}^{-1} \boldsymbol{X}' \Sigma^{-1} \boldsymbol{S} [\boldsymbol{S}' (\boldsymbol{M}_X \Sigma \boldsymbol{M}_X)^+ \boldsymbol{S}]^{-1} \boldsymbol{U}' \\ -[\boldsymbol{S}' (\boldsymbol{M}_X \Sigma \boldsymbol{M}_X)^+ \boldsymbol{S}]^{-1} \boldsymbol{U}' \end{pmatrix}.$$

Let us (for the sake of simplicity) use the notation $\boldsymbol{Z} = \boldsymbol{S}' (\boldsymbol{M}_X \Sigma \boldsymbol{M}_X)^+ \boldsymbol{S}$, then

$$\begin{pmatrix} \hat{\hat{\beta}} \\ \hat{\hat{\kappa}} \end{pmatrix} = \left\{ \begin{pmatrix} \boldsymbol{I}, & \boldsymbol{0} \\ \boldsymbol{0}, & \boldsymbol{I} \end{pmatrix} - \begin{pmatrix} \boldsymbol{C}^{-1} \boldsymbol{B}_1' + \boldsymbol{C}^{-1} \boldsymbol{X}' \Sigma^{-1} \boldsymbol{S} \boldsymbol{Z}^{-1} \boldsymbol{U}' \\ -\boldsymbol{Z}^{-1} \boldsymbol{U}' \end{pmatrix} \right.$$

$$\left. \times [\boldsymbol{B}_1 \boldsymbol{C}^{-1} \boldsymbol{B}_1' + \boldsymbol{U} \boldsymbol{Z}^{-1} \boldsymbol{U}']^{-1} (\boldsymbol{B}_1, \boldsymbol{B}_2) \right\} \begin{pmatrix} \hat{\beta} \\ \hat{\kappa} \end{pmatrix}$$

$$- \begin{pmatrix} \boldsymbol{C}^{-1} \boldsymbol{B}_1' + \boldsymbol{C}^{-1} \boldsymbol{X}' \Sigma^{-1} \boldsymbol{S} \boldsymbol{Z}^{-1} \boldsymbol{U}' \\ -\boldsymbol{Z}^{-1} \boldsymbol{U}' \end{pmatrix} [\boldsymbol{B}_1 \boldsymbol{C}^{-1} \boldsymbol{B}_1' + \boldsymbol{U} \boldsymbol{Z}^{-1} \boldsymbol{U}']^{-1} \boldsymbol{b}.$$

Thus

$$
\begin{aligned}
\hat{\hat{\beta}} = & \left\{ \boldsymbol{I} - (\boldsymbol{C}^{-1}\boldsymbol{B}_1' + \boldsymbol{C}^{-1}\boldsymbol{X}'\Sigma^{-1}\boldsymbol{S}\boldsymbol{Z}^{-1}\boldsymbol{U}')[\boldsymbol{B}_1\boldsymbol{C}^{-1}\boldsymbol{B}_1' + \boldsymbol{U}\boldsymbol{Z}^{-1}\boldsymbol{U}']^{-1}\boldsymbol{B}_1 \right\} \hat{\beta} \\
& - (\boldsymbol{C}^{-1}\boldsymbol{B}_1' + \boldsymbol{C}^{-1}\boldsymbol{X}'\Sigma^{-1}\boldsymbol{S}\boldsymbol{Z}^{-1}\boldsymbol{U}')[\boldsymbol{B}_1\boldsymbol{C}^{-1}\boldsymbol{B}_1' + \boldsymbol{U}\boldsymbol{Z}^{-1}\boldsymbol{U}']^{-1}\boldsymbol{B}_2\hat{\kappa} \\
& - (\boldsymbol{C}^{-1}\boldsymbol{B}_1' + \boldsymbol{C}^{-1}\boldsymbol{X}'\Sigma^{-1}\boldsymbol{S}\boldsymbol{Z}^{-1}\boldsymbol{U}')[\boldsymbol{B}_1\boldsymbol{C}^{-1}\boldsymbol{B}_1' + \boldsymbol{U}\boldsymbol{Z}^{-1}\boldsymbol{U}']^{-1}\boldsymbol{b}.
\end{aligned}
$$

$$
\begin{aligned}
\hat{\hat{\kappa}} = & \boldsymbol{Z}^{-1}\boldsymbol{U}'[\boldsymbol{B}_1\boldsymbol{C}^{-1}\boldsymbol{B}_1' + \boldsymbol{U}\boldsymbol{Z}^{-1}\boldsymbol{U}']^{-1}\boldsymbol{B}_1\hat{\beta} \\
& + \left[\boldsymbol{I} + \boldsymbol{Z}^{-1}\boldsymbol{U}'(\boldsymbol{B}_1\boldsymbol{C}^{-1}\boldsymbol{B}_1' + \boldsymbol{U}\boldsymbol{Z}^{-1}\boldsymbol{U}')^{-1}\boldsymbol{B}_2 \right] \hat{\kappa} \\
& + \boldsymbol{Z}^{-1}\boldsymbol{U}'[\boldsymbol{B}_1\boldsymbol{C}^{-1}\boldsymbol{B}_1' + \boldsymbol{U}\boldsymbol{Z}^{-1}\boldsymbol{U}']^{-1}\boldsymbol{b}.
\end{aligned}
$$

The statement of the Theorem 3 is now obvious. □

Theorem 4 *For the BLUEs* $\hat{\hat{\beta}}$, $\hat{\hat{\kappa}}$ *it is valid*

$$
\begin{aligned}
\operatorname{var}(\hat{\hat{\beta}}) = & \operatorname{var}(\hat{\beta}) - (\boldsymbol{C}^{-1}\boldsymbol{B}_1' + \boldsymbol{C}^{-1}\boldsymbol{X}'\Sigma^{-1}\boldsymbol{S}\boldsymbol{Z}^{-1}\boldsymbol{U}')[\boldsymbol{B}_1\boldsymbol{C}^{-1}\boldsymbol{B}_1' + \boldsymbol{U}\boldsymbol{Z}^{-1}\boldsymbol{U}']^{-1} \\
& \times (\boldsymbol{B}_1\boldsymbol{C}^{-1} + \boldsymbol{U}\boldsymbol{Z}^{-1}\boldsymbol{S}'\Sigma^{-1}\boldsymbol{X}\boldsymbol{C}^{-1}), \qquad (11)
\end{aligned}
$$

$$
\operatorname{var}(\hat{\hat{\kappa}}) = \operatorname{var}(\hat{\kappa}) - \boldsymbol{Z}^{-1}\boldsymbol{U}'[\boldsymbol{B}_1\boldsymbol{C}^{-1}\boldsymbol{B}_1' + \boldsymbol{U}\boldsymbol{Z}^{-1}\boldsymbol{U}']^{-1}\boldsymbol{U}\boldsymbol{Z}^{-1}. \qquad (12)
$$

Proof We have

$$
\operatorname{var}(\hat{\hat{\beta}}) = \operatorname{var}[\boldsymbol{A}\hat{\beta} - \boldsymbol{B}\hat{\kappa}],
$$

where

$$
\begin{aligned}
\boldsymbol{A} &= \boldsymbol{I} - (\boldsymbol{C}^{-1}\boldsymbol{B}_1' + \boldsymbol{C}^{-1}\boldsymbol{X}'\Sigma^{-1}\boldsymbol{S}\boldsymbol{Z}^{-1}\boldsymbol{U}')[\boldsymbol{B}_1\boldsymbol{C}^{-1}\boldsymbol{B}_1' + \boldsymbol{U}\boldsymbol{Z}^{-1}\boldsymbol{U}']^{-1}\boldsymbol{B}_1, \\
\boldsymbol{B} &= (\boldsymbol{C}^{-1}\boldsymbol{B}_1' + \boldsymbol{C}^{-1}\boldsymbol{X}'\Sigma^{-1}\boldsymbol{S}\boldsymbol{Z}^{-1}\boldsymbol{U}')[\boldsymbol{B}_1\boldsymbol{C}^{-1}\boldsymbol{B}_1' + \boldsymbol{U}\boldsymbol{Z}^{-1}\boldsymbol{U}']^{-1}\boldsymbol{B}_2.
\end{aligned}
$$

Analologously

$$
\operatorname{var}(\hat{\hat{\kappa}}) = \operatorname{var}[\boldsymbol{F}\hat{\beta} + \boldsymbol{G}\hat{\kappa}],
$$

where

$$
\begin{aligned}
\boldsymbol{F} &= \boldsymbol{Z}^{-1}\boldsymbol{U}'[\boldsymbol{B}_1\boldsymbol{C}^{-1}\boldsymbol{B}_1' + \boldsymbol{U}\boldsymbol{Z}^{-1}\boldsymbol{U}']^{-1}\boldsymbol{B}_1, \\
\boldsymbol{G} &= \boldsymbol{I} + \boldsymbol{Z}^{-1}\boldsymbol{U}'[\boldsymbol{B}_1\boldsymbol{C}^{-1}\boldsymbol{B}_1' + \boldsymbol{U}\boldsymbol{Z}^{-1}\boldsymbol{U}']^{-1}\boldsymbol{B}_2.
\end{aligned}
$$

We get the expressions for $\operatorname{var}(\hat{\hat{\beta}})$ and $\operatorname{var}(\hat{\hat{\kappa}})$ after longer but easy calculations. □

Example 1 Consider the following situation. Let's have points F_1, F_2 and F_3 of existing local network and points P_1 and P_2, for which it is necessary to estimate their coordinates (see Figure 1). We have the measured values Y_1, Y_2 of coordinates of the point $F_1 = (\beta_1, \beta_2)$, the measured values Y_3, Y_4 of coordinates of the point $F_2 = (\beta_3, \beta_4)$ and the measured values Y_5, Y_6 of coordinates of the point $F_3 = (\beta_5, \beta_6)$. Moreover, we have the measured values Y_7, Y_8, Y_9, Y_{10} and Y_{11} of angles β_7 and β_8 and distances β_9, β_{10} and β_{11}. Finally, we know the measured values Y_{12} and Y_{13} of angles κ_1 and κ_2. The values β and κ are in meters and in radians, respectively.

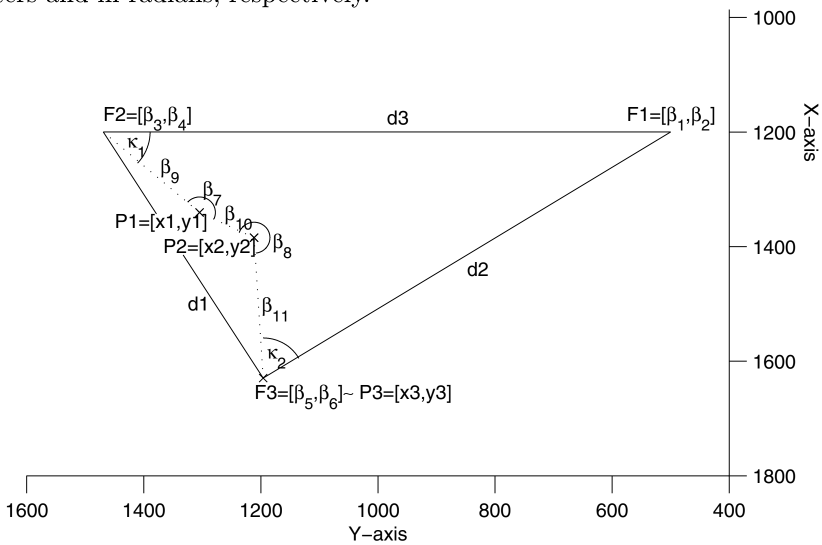

Figure 1: Layout of the situation in Example 1

We have the model (1), where $(X, S) = I_{13}$.

Assume the results of measurements to be (see [4])

$$\mathbf{Y} = \begin{pmatrix} 1200.003 \text{ m} \\ 499.999 \text{ m} \\ 1200.001 \text{ m} \\ 1469.113 \text{ m} \\ 1629.649 \text{ m} \\ 1196.073 \text{ m} \\ 2.876604026 \text{ rad} \\ 4.207717253 \text{ rad} \\ 216.347 \text{ m} \\ 103.095 \text{ m} \\ 245.478 \text{ m} \\ 0.707031134 \text{ rad} \\ 1.080434554 \text{ rad} \end{pmatrix}.$$

We take the covariance matrix Σ from the model (1) in the form

$$\Sigma = \begin{pmatrix} \Sigma^F & 0_{6,5} & 0_{6,2} \\ 0_{5,6} & \Sigma^{d,a} & 0_{5,2} \\ 0_{2,6} & 0_{2,5} & \Sigma^a \end{pmatrix}.$$

We assume the coordinate accuracy of the points F_1, F_2 and F_3 of existing local network to be approximately the same as the accuracy of measured parameters β_j, $j = 7, \ldots, 11$, and as the accuracy of measured parameters κ_1 and κ_2.

The accuracy of coordinates Y_i, $i = 1 \dots 6$, of the points F_1, F_2 and F_3 is given by the covariance matrix Σ^F:

$$\Sigma^F = 0.001^2 \times \begin{pmatrix} 1.6987 & 1.5583 & 0.1928 & 1.0711 & -1.8915 & -2.6295 \\ 1.5583 & 7.3592 & -1.4785 & -3.895 & -0.0798 & -3.4642 \\ 0.1928 & -1.4785 & 5.0406 & -1.4122 & -5.2334 & 2.8907 \\ 1.0711 & -3.895 & -1.4122 & 6.5277 & 0.341 & -2.6328 \\ -1.8915 & -0.0798 & -5.2334 & 0.341 & 7.125 & -0.2613 \\ -2.6295 & -3.4642 & 2.8907 & -2.6328 & -0.2613 & 6.097 \end{pmatrix}.$$

The accuracy of measured distances was 3 mm and the accuracy of measured angles was 5 cc $= 5\pi/(200 \cdot 100 \cdot 100) = 5/636620$, (the standard deviation of the theodolite is $\sigma_t = 5$ cc, i.e. that which corresponds to 5 centesimal seconds). We thus suppose that the covariance matrix for $(Y_7, \dots, Y_{11})$ is

$$\begin{aligned} \Sigma^{d,a} &= \begin{pmatrix} 0.003^2 \times \boldsymbol{I}_{3,3} & \boldsymbol{0}_{3,2} \\ \boldsymbol{0}_{2,3} & (\frac{5\pi}{200\cdot 100\cdot 100})^2 \times \boldsymbol{I}_{2,2} \end{pmatrix} \\ &= \begin{pmatrix} 0.003^2 \times \boldsymbol{I}_{3,3} & \boldsymbol{0}_{3,2} \\ \boldsymbol{0}_{2,3} & 6.17 \times 10^{-11} \times \boldsymbol{I}_{2,2} \end{pmatrix}. \end{aligned}$$

Accordingly, we suppose that the covariance matrix of measured angles $(\boldsymbol{Y}_{12}, \boldsymbol{Y}_{13})$ is

$$\Sigma^a = \left(\frac{5\pi}{200 \cdot 100 \cdot 100}\right)^2 \times \boldsymbol{I}_{2,2} = \left(\frac{5}{636620}\right)^2 \times \boldsymbol{I}_{2,2} = 6.17 \times 10^{-11} \times \boldsymbol{I}_{2,2}\,.$$

The aim is to find conditions for parameters β and κ.

To that end, we first determine (see Figure 1) the coordinates of points $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$ and $P_3 = (x_3, y_3)$:

$$\begin{aligned} x_1 &= \beta_3 + \beta_9 \cos\left(\tfrac{\pi}{2} + \kappa_1\right), \\ y_1 &= \beta_4 + \beta_9 \cos\left(\tfrac{\pi}{2} + \kappa_1\right), \end{aligned}$$

(it follows from the fact that the point P_1 shall be situated on a circle with circumference β_9 and with center in point F_2, and from the fact that the point P_1 is reached from the point F_2 via the angle $\angle F_1, F_2, P_1 = \kappa_1$);

$$\begin{aligned} x_2 &= x_1 + \beta_{10} \cos\left(\left(\arctan \tfrac{\beta_4 - y_1}{\beta_3 - x_1} + 0 \cdot \pi\right) + \pi + \beta_7\right), \\ y_2 &= y_1 + \beta_{10} \sin\left(\left(\arctan \tfrac{\beta_4 - y_1}{\beta_3 - x_1} + 0 \cdot \pi\right) + \pi + \beta_7\right), \end{aligned}$$

(it follows from the fact that the point P_2 shall be situated on a circle with circumference β_{10} and with center in point P_1, and from the fact that the point P_2 is reached from the point P_1 via the angle $\angle F_2, P_1, P_2 = \beta_7$);

$$\begin{aligned} x_3 &= x_2 + \beta_{11} \cos\left(\left(\arctan \tfrac{y_1 - y_2}{x_1 - x_2} + 0 \cdot \pi\right) + \pi + \beta_8\right), \\ y_3 &= y_2 + \beta_{11} \sin\left(\left(\arctan \tfrac{y_1 - y_2}{x_1 - x_2} + 0 \cdot \pi\right) + \pi + \beta_8\right), \end{aligned}$$

(it follows from the fact that the point P_3 shall be situated on a circle with circumference β_{11} and with center in point P_2, and from the fact that the point P_3 is reached from the point P_2 via the angle $\angle P_1, P_2, P_3 = \beta_8$).

It can be seen from Figure 1 that the conditions $\mathbf{g}(\beta, \kappa) = \boldsymbol{o}$ for parameters β and κ are (involving the conditions given above)

$$\begin{aligned} g_1 &= (x_3 - \beta_5)^2 + (y_3 - \beta_6)^2 = 0, \\ g_2 &= \left(\pi + \arctan \frac{y_3 - \beta_2}{x_3 - \beta_1}\right) - \left(\pi + \arctan \frac{y_2 - y_3}{x_2 - x_3}\right) - \kappa_2 = 0. \end{aligned}$$

The first constraint says that the point P_3 is equivalent to F_3.

The second constraint reflects the fact that $\angle P_2, P_3, F_1 = \kappa_2$.

Now we use the Taylor expansion—the linear version of the condition

$$\mathbf{g}(\beta, \kappa) = \begin{pmatrix} g_1(\beta, \kappa) \\ g_2(\beta, \kappa) \end{pmatrix} = \boldsymbol{o}$$

is $\boldsymbol{B}_1 \delta\beta + \boldsymbol{B}_2 \delta\kappa + b = \boldsymbol{o}$, where the matrix $\boldsymbol{B}_1 = \frac{\partial \mathbf{g}(\beta^0, \kappa^0)}{\partial \beta'}$, $\boldsymbol{B}_2 = \frac{\partial \mathbf{g}(\beta^0, \kappa^0)}{\partial \kappa'}$, and $b = \mathbf{g}(\beta^0, \kappa^0)$ at the approximate point.

So we can consider the model (8).

In the linearized model we determine numerically the estimates and the covariance matrices according to Theorem 3 and Theorem 4

$$\hat{\hat{\beta}} = \begin{pmatrix} 1200.000 \text{ m} \\ 500.000 \text{ m} \\ 1200.000 \text{ m} \\ 1469.112 \text{ m} \\ 1629.651 \text{ m} \\ 1196.073 \text{ m} \\ 2.876605771 \text{ rad} \\ 4.207720046 \text{ rad} \\ 216.347 \text{ m} \\ 103.096 \text{ m} \\ 245.475 \text{ m} \end{pmatrix} \quad \text{and} \quad \hat{\hat{\kappa}} = \begin{pmatrix} 0.707030785 \text{ rad} \\ 1.080438743 \text{ rad} \end{pmatrix}.$$

$$\text{var}(\hat{\hat{\beta}}) = \left(Q_1 \,,\, Q_2 \right) \quad \text{and} \quad \text{var}(\hat{\hat{\kappa}}) = \begin{pmatrix} 4.90 \cdot 10^{-11} & -9.24 \cdot 10^{-12} \\ -9.24 \cdot 10^{-12} & 4.30 \cdot 10^{-11} \end{pmatrix},$$

where

$$Q_1 = \begin{pmatrix} 1.43 \cdot 10^{-6} & 1.79 \cdot 10^{-6} & -4.45 \cdot 10^{-7} & 8.08 \cdot 10^{-7} & -9.83 \cdot 10^{-7} & -2.60 \cdot 10^{-6} \\ 1.79 \cdot 10^{-6} & 7.14 \cdot 10^{-6} & -1.01 \cdot 10^{-6} & -3.60 \cdot 10^{-6} & -7.73 \cdot 10^{-7} & -3.54 \cdot 10^{-6} \\ -4.45 \cdot 10^{-7} & -1.01 \cdot 10^{-6} & 3.34 \cdot 10^{-6} & -1.83 \cdot 10^{-6} & -2.90 \cdot 10^{-6} & 2.85 \cdot 10^{-6} \\ 8.08 \cdot 10^{-7} & -3.60 \cdot 10^{-6} & -1.83 \cdot 10^{-6} & 6.07 \cdot 10^{-6} & 1.02 \cdot 10^{-6} & -2.47 \cdot 10^{-6} \\ -9.83 \cdot 10^{-7} & -7.73 \cdot 10^{-7} & -2.90 \cdot 10^{-6} & 1.02 \cdot 10^{-6} & 3.88 \cdot 10^{-6} & -2.49 \cdot 10^{-7} \\ -2.60 \cdot 10^{-6} & -3.54 \cdot 10^{-6} & 2.85 \cdot 10^{-6} & -2.47 \cdot 10^{-6} & -2.49 \cdot 10^{-7} & 6.01 \cdot 10^{-6} \\ 6.18 \cdot 10^{-10} & -1.03 \cdot 10^{-9} & 1.19 \cdot 10^{-10} & 1.96 \cdot 10^{-9} & -7.29 \cdot 10^{-10} & -9.31 \cdot 10^{-10} \\ 1.03 \cdot 10^{-9} & -1.32 \cdot 10^{-9} & 1.23 \cdot 10^{-9} & 2.21 \cdot 10^{-9} & -2.26 \cdot 10^{-9} & -8.86 \cdot 10^{-10} \\ -2.36 \cdot 10^{-7} & -8.35 \cdot 10^{-8} & -1.30 \cdot 10^{-6} & 5.29 \cdot 10^{-7} & 1.54 \cdot 10^{-6} & -4.45 \cdot 10^{-7} \\ 1.95 \cdot 10^{-7} & -3.96 \cdot 10^{-7} & -1.52 \cdot 10^{-7} & 8.10 \cdot 10^{-7} & -4.31 \cdot 10^{-8} & -4.13 \cdot 10^{-7} \\ -1.32 \cdot 10^{-6} & 8.08 \cdot 10^{-7} & -3.93 \cdot 10^{-6} & -4.53 \cdot 10^{-7} & 5.26 \cdot 10^{-6} & -3.56 \cdot 10^{-7} \end{pmatrix},$$

$$Q_2 = \begin{pmatrix} 6.18\cdot10^{-10} & 1.03\cdot10^{-9} & -2.36\cdot10^{-7} & 1.95\cdot10^{-7} & -1.32\cdot10^{-6}\\ -1.03\cdot10^{-9} & -1.32\cdot10^{-9} & -8.35\cdot10^{-8} & -3.96\cdot10^{-7} & 8.08\cdot10^{-7}\\ 1.11\cdot10^{-10} & 1.23\cdot10^{-9} & -1.30\cdot10^{-6} & -1.52\cdot10^{-7} & -3.93\cdot10^{-6}\\ 1.96\cdot10^{-9} & 2.21\cdot10^{-9} & 5.29\cdot10^{-7} & 8.10\cdot10^{-7} & -4.53\cdot10^{-7}\\ -7.29\cdot10^{-10} & -2.26\cdot10^{-9} & 1.54\cdot10^{-6} & -4.31\cdot10^{-8} & 5.26\cdot10^{-6}\\ -9.31\cdot10^{-10} & -8.86\cdot10^{-10} & -4.45\cdot10^{-7} & -4.13\cdot10^{-7} & -3.56\cdot10^{-7}\\ 5.11\cdot10^{-11} & -1.05\cdot10^{-11} & -4.57\cdot10^{-9} & -4.63\cdot10^{-9} & -2.57\cdot10^{-9}\\ -1.05\cdot10^{-11} & 5.05\cdot10^{-11} & -3.63\cdot10^{-9} & -4.45\cdot10^{-9} & 6.83\cdot10^{-11}\\ -4.57\cdot10^{-9} & -3.63\cdot10^{-9} & 2.20\cdot10^{-5} & -2.15\cdot10^{-6} & -4.26\cdot10^{-6}\\ -4.63\cdot10^{-9} & -4.45\cdot10^{-9} & -2.15\cdot10^{-6} & 2.30\cdot10^{-5} & -1.59\cdot10^{-6}\\ -2.57\cdot10^{-9} & 6.83\cdot10^{-11} & -4.26\cdot10^{-6} & -1.59\cdot10^{-6} & 1.51\cdot10^{-5} \end{pmatrix}.$$

All computations in the example were performed in Matlab.

To make comparisons easier, the following table shows the results.

$\boldsymbol{Y}_{[i]}$	$\sqrt{\mathrm{var}(\boldsymbol{Y})}_{[i,i]}$	$\begin{pmatrix}\hat{\hat{\beta}}\\ \hat{\hat{\kappa}}\end{pmatrix}_{[i]}$	$\sqrt{\left[\mathrm{var}\begin{pmatrix}\hat{\hat{\beta}}\\ \hat{\hat{\kappa}}\end{pmatrix}\right]_{[i,i]}}$	$\boldsymbol{Y}_{[i]}-\begin{pmatrix}\hat{\hat{\beta}}\\ \hat{\hat{\kappa}}\end{pmatrix}_{[i]}$
1200.003 m	4.12 mm	1200.000 m	1.13 mm	3 mm
499.999 m	8.58 mm	500.000 m	2.67 mm	−1 mm
1200.001 m	7.10 mm	1200.000 m	1.83 mm	1 mm
1469.113 m	8.08 mm	1469.112 m	2.46 mm	1 mm
1629.649 m	8.44 mm	1629.651 m	1.97 mm	−2 mm
1196.073 m	7.81 mm	1196.073 m	2.45 mm	0 mm
2.876604026 rad	5.00 cc	2.87605771 rad	4.55 cc	−1.111 cc
4.207717253 rad	5.00 cc	4.20772005 rad	4.53 cc	−1.778 cc
216.347 m	3.00 mm	216.347 m	4.69 mm	0 mm
103.095 m	3.00 mm	103.096 m	4.79 mm	−1 mm
245.478 m	3.00 mm	245.475 m	3.89 mm	3 mm
0.707031134 rad	5.00 cc	0.707030785 rad	4.46 cc	0.222 cc
1.080434554 rad	5.00 cc	1.080438743 rad	4.17 cc	−2.667 cc

The second column shows that the dispersions of elements of the measured vector $\boldsymbol{Y}$ are different. We can see in the table that dispersions of some elements of estimators $\hat{\hat{\beta}}$ and $\hat{\hat{\kappa}}$ have decreased and some have increased in the process of estimation, which is due to the tendency to distribute the uncertainty of measurements equally.

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On the Existence of One-Signed Periodic Solutions of Some Differential Equations of Second Order

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Abstract

We study the existence of one-signed periodic solutions of the equations

$$x''(t) - a^2(t)x(t) + \mu f(t, x(t), x'(t)) = 0,$$
$$x''(t) + a^2(t)x(t) = \mu f(t, x(t), x'(t)),$$

where $\mu > 0$, $a : (-\infty, +\infty) \to (0, \infty)$ is continuous and 1-periodic, f is a continuous and 1-periodic in the first variable and may take values of different signs. The Krasnosielski fixed point theorem on cone is used.

Key words: Positive solutions; boundary value problems; cone; fixed point theorem.

2000 Mathematics Subject Classification: 34G20, 34K10, 34B10, 34B15

1 Introduction

Nonnegative solutions to varius boundary value problems for ordinary differential equations have been considered by several authors (see for instance in

[1]–[8]). This paper deals with existence of positive (negative) periodic solutions of the nonlinear differential equations of the form

$$x''(t) - a^2(t)x(t) + \mu f(t, x(t), x'(t)) = 0, \tag{1.1}$$

$$x''(t) + a^2(t)x(t) = \mu f(t, x(t), x'(t)), \tag{1.2}$$

where $a : (-\infty, +\infty) \to (0, \infty)$ is continuous, 1-periodic, $\mu > 0$, f is a continuous, 1-periodic function in t and may take values of different signs. Existence in this paper will be established using Krasnosielski fixed point theorem in a cone, which we state here for the convenience of the reader.

Theorem 1.1 (K. Deimling [4], D. Guo, V. Laksmikannthan [5]). *Let $E = (E, \|\cdot\|)$ be a Banach space and let $K \subset E$ be a cone in E. Assume Ω_1 and Ω_2 are bounded and open subsets of E with $0 \in \Omega_1$ and $\overline{\Omega_1} \subset \Omega_2$ and let $A : K \cap (\overline{\Omega_2} \setminus \Omega_1) \to K$ be continuous and completely continuous. In addition suppose either $\|Au\| \le \|u\|$ for $u \in K \cap \partial\Omega_1$ and $\|Au\| \ge \|u\|$ for $K \cap \partial\Omega_2$ or $\|Au\| \ge \|u\|$ for $u \in K \cap \partial\Omega_1$ and $\|Au\| \le \|u\|$ for $u \in K \cap \partial\Omega_2$ hold. Then A has a fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.*

2 Preliminary results

First, we shall give some notation. We define $P_1^m(\mathbb{R})$ $(m \in \mathbb{N})$ to be the subspace of BC($\mathbb{R}$) (bounded, continuous real functions on $\mathbb{R}$) consisting of all 1-periodic mapping x such that $x^{(m)}$ is an 1-periodic and continuous function on $\mathbb{R}$. For $x \in P_1^1(\mathbb{R})$ we define

$$\|x\|_1 = \sup_{t \in [0,1]} [|x(t)| + |x'(t)|].$$

Note $P_1^1(\mathbb{R}, \|\cdot\|_1)$ is a Banach space.

Let us consider the boundary value problems

$$-\left(x''(t) - a^2(t)x(t)\right) = 0, \quad x(0) = x(1),\ x'(0) = x'(1); \tag{2.1}$$

$$x''(t) + a^2(t)x(t) = 0, \quad x(0) = x(1),\ x'(0) = x'(1), \tag{2.2}$$

In this paper we assume conditions under which the only solution of the problem (2.1) or (2.2) is the trivial one. In the proofs of theorems we will make use the Green functions G_1 and G_2 of the boundary value problems (2.1) and (2.2).

Remark 2.1 If $a \in C[0,1]$ and $a(t) > 0$ for all $t \in [0,1]$, then the problem (2.1) has only the trivial solution and $G_1(t,s) > 0$ for all $t, s \in [0,1]$ (see [7]).

If $a \in C[0,1],\ a(t) > 0$ for $t \in [0,1]$ and $\sup_{t\in[0,1]} a(t) < \pi$, then the problem (2.2) has only the trivial solution and $G_2(t,s) > 0$ for all $t, s \in [0,1]$ (see [7]).

Remark 2.2 If $a(t) \equiv k > 0$ for $t \in [0,1]$, then

$$G_1(t,s) = \frac{1}{2k(e^k - 1)} \begin{cases} e^{k(1-s+t)} + e^{k(s-t)}, & 0 \le t \le s \le 1 \\ e^{k(t-s)} + e^{k(1+s-t)}, & 0 \le s \le t \le 1. \end{cases}$$

Remark 2.3 If $a(t) \equiv k > 0$ for $t \in [0,1]$ and $k \neq 2l\pi$ for all $l \in \mathbb{N}$, then

$$G_2(t,s) = \frac{1}{2k \sin k/2} \cos k[1/2 - |s-t|].$$

Before giving the lemmas we shall introduce some notation. We denote

$$\overline{M_i} = \sup_{t,s\in[0,1]} G_i(t,s), \qquad \overline{m_i} = \inf_{t,s\in[0,1]} G_i(t,s),$$

$$\overline{M_i}' = \sup_{t,s\in[0,1]} \left| \frac{\partial G_i}{\partial t}(t,s) \right|, \qquad \overline{m_i}' = \inf_{t,s\in[0,1]} \left| \frac{\partial G_i}{\partial t}(t,s) \right|$$

for $i = 1,2$.

The properties of the functions G_i $(i = 1,2)$ needed later on are described by the following lemmas.

Lemma 2.4 *Suppose that*

(2.3) $\qquad f : \mathbb{R}^3 \to \mathbb{R}$ *is continuous,* $a \in C[0,1]$ *and* $a(t) > 0$ *for* $t \in [0,1]$.

Then $x \in C^2[0,1]$ *is a solution of the problem*

$$\begin{cases} x''(t) - a^2(t)x(t) + \mu f(t,x(t),x'(t)) = 0 \\ x(0) = x(1), \quad x'(0) = x'(1) \end{cases} \tag{2.4}$$

if and only if x *satisfies the integral equation*

$$x(t) = \mu \int_0^1 G_1(t,s) f(s,x(s),x'(s))\, ds. \tag{2.5}$$

Lemma 2.5 *Suppose that* $a \in C[0,1], a(t) > 0$ *for* $t \in [0,1]$, $\sup_{t\in[0,1]} a(t) < \pi$ *and* $f : \mathbb{R}^3 \to \mathbb{R}$ *is continuous. Then* $x \in C^2[0,1]$ *is a solution of the problem*

$$\begin{cases} x''(t) + a^2(t)x(t) = \mu f(t,x(t),x'(t)) \\ x(0) = x(1), \quad x'(0) = x'(1) \end{cases} \tag{2.6}$$

if and only if x *satisfies the integral equation*

$$x(t) = \mu \int_0^1 G_2(t,s) f(s,x(s),x'(s))\, ds. \tag{2.7}$$

Lemma 2.6 *Let* $a \in C[0,1]$ *and* $a(t) > 0$ *for* $t \in [0,1]$. *Then*

$$\inf_{t,s\in[0,1]} G_1(t,s) = \inf_{t\in[0,1]} G_1(t,1), \quad \text{(see [7])} \tag{2.8}$$

$$\begin{cases} d_0 G_1(t,s) - \left|\frac{\partial G_1}{\partial t}(t,s)\right| \geq G_1(s,s) + \left|\frac{\partial G_1}{\partial t}(s-0,s)\right| \\ \text{for } s,t \in [0,1] \text{ and} \\ d_0 G_1(t,s) - \left|\frac{\partial G_1}{\partial t}(t,s)\right| \geq G_1(s,s) + \left|\frac{\partial G_1}{\partial t}(s+0,s)\right| \\ \text{for } s,t \in [0,1] \text{ where } \frac{\partial G_1}{\partial t}(s-0,s) \quad \left(\frac{\partial G_1}{\partial t}(s+0,s)\right) \\ \text{denote the left-hand (the right-hand) side derivative of } G_1 \\ \text{at the point } (s,s) \text{ and } d_0 \geq \frac{2\overline{M_1}' + \overline{M_1}}{\overline{m_1}}, \end{cases} \tag{2.9}$$

$$(2.10)\qquad G_1(s,s)+\left|\frac{\partial G_1}{\partial t}(s-0,s)\right|\geq M_0\left(G_1(t,s)+\left|\frac{\partial G_1}{\partial t}(t,s)\right|\right)$$

for $s,t\in[0,1]$ *and* $M_0\in\left(0,\frac{\overline{m_1}+\overline{m_1}'}{\overline{M_1}+\overline{M_1}'}\right]$,

$$(2.11)\qquad G_1(s,s)+\left|\frac{\partial G_1}{\partial t}(s+0,s)\right|\geq M_0\left(G_1(t,s)+\left|\frac{\partial G_1}{\partial t}(t,s)\right|\right),$$

where $s,t\in[0,1]$.

Lemma 2.7 *Let* $a\in C[0,1]$ *and* $a(t)>0$ *for* $t\in[0,1]$ *and* $\sup_{t\in[0,1]}a(t)<\pi$. *Then*

$$(2.12)\qquad \sup_{t,s\in[0,1]}G_2(t,s)=\sup_{t\in[0,1]}G_2(t,1)\quad\text{(see [7])},$$

$$(2.13)\qquad \overline{d_0}G_2(t,s)-\left|\frac{\partial G_2}{\partial t}(t,s)\right|\geq G_2(s,s)+\left|\frac{\partial G_2}{\partial t}(s-0,s)\right|$$

for $t,s\in[0,1]$ *and* $\overline{d_0}\geq\frac{2\overline{M_2}'+\overline{M_2}}{\overline{m_2}}$,

$$(2.14)\qquad \overline{d_0}G_2(t,s)-\left|\frac{\partial G_2}{\partial t}(t,s)\right|\geq G_2(s,s)+\left|\frac{\partial G_2}{\partial t}(s+0,s)\right|,$$

where $s,t\in[0,1]$,

$$(2.15)\qquad G_2(s,s)+\left|\frac{\partial G_2}{\partial t}(s-0,s)\right|\geq\overline{M_0}\left(G_2(t,s)+\left|\frac{\partial G_2}{\partial t}(t,s)\right|\right)$$

for $s,t\in[0,1]$, $\overline{M_0}\in\left(0,\frac{\overline{m_2}+\overline{m_2}'}{\overline{M_2}+\overline{M_2}'}\right]$ *and*

$$(2.16)\qquad G_2(s,s)+\left|\frac{\partial G_2}{\partial t}(s+0,s)\right|\geq\overline{M_0}\left(G_2(t,s)+\left|\frac{\partial G_2}{\partial t}(t,s)\right|\right),$$

where $s,t\in[0,1]$.

It is not difficult to prove the following

Corollary 2.8 *Let* $a(t)\equiv k>0$ *for* $t\in[0,1]$. *Then*

$$(2.8)'\quad\begin{cases}\sup_{t,s\in[0,1]}G_1(t,s)=\frac{e^k+1}{2k(e^k-1)},\\ \inf_{t,s\in[0,1]}G_1(t,s)=\frac{e^{k/2}}{k(e^k-1)},\\ G_1(s,s)\geq G_1(t,s)\ \text{ for } s,t\in[0,1],\quad \sup_{t,s\in[0,1]}\left|\frac{\partial G_1}{\partial t}(t,s)\right|=\frac{1}{2},\\ \inf_{t,s\in[0,1]}\left|\frac{\partial G_1}{\partial t}(t,s)\right|=0,\quad \int_0^1G_1(t,s)\,ds=\frac{1}{k^2}\ \text{ for } t\in[0,1],\\ \sup_{t\in[0,1]}\int_0^1G_1(t,s)\,ds+\sup_{t\in[0,1]}\int_0^1\left|\frac{\partial G_1}{\partial t}(t,s)\right|ds=m_1\leq\frac{1}{k^2}+\frac{1}{2},\end{cases}$$

$$(2.9)' \qquad \begin{cases} d_0 G_1(t,s) - \left|\frac{\partial G_1}{\partial t}(t,s)\right| \geq G_1(s,s) + \left|\frac{\partial G_1}{\partial t}(s-0,s)\right| \\ \text{for } s,t \in [0,1] \text{ and} \\ d_0 G_1(t,s) - \left|\frac{\partial G_1}{\partial t}(t,s)\right| \geq G_1(s,s) + \left|\frac{\partial G_1}{\partial t}(s+0,s)\right| \\ \text{for } s,t \in [0,1] \text{ and } d_0 \geq \frac{e^k+1+2k(e^k-1)}{2e^{k/2}}, \end{cases}$$

$$(2.10)' \qquad G_1(s,s) + \left|\frac{\partial G_1}{\partial t}(s-0,s)\right| \geq M_0 \left(G_1(t,s) + \left|\frac{\partial G_1}{\partial t}(t,s)\right|\right)$$

for $s,t \in [0,1]$ *and* $M_0 \in \left(0, \frac{2e^{k/2}}{e^k(1+k)+1-k}\right]$,

$$(2.11)' \qquad G_1(s,s) + \left|\frac{\partial G_1}{\partial t}(s+0,s)\right| \geq M_0 \left(G_1(t,s) + \left|\frac{\partial G_1}{\partial t}(t,s)\right|\right),$$

where $s,t \in [0,1]$.

Corollary 2.9 *Let* $a(t) \equiv k$ *for* $t \in [0,1]$ *and let* $0 < k < \pi$. *Then*

$$(2.12)' \qquad \begin{cases} \inf_{t,s\in[0,1]} G_2(t,s) = \frac{\cot k/2}{2k}, \\ \sup_{t,s\in[0,1]} G_2(t,s) = \frac{1}{2k \sin^{k/2}}, \quad \sup_{t,s\in[0,1]} \left|\frac{\partial G_2}{\partial t^{(t,s)}}(t,s)\right| = \frac{1}{2}, \\ \inf_{t,s\in[0,1]} \left|\frac{\partial G_2}{\partial t}(t,s)\right| = 0, \quad \int_0^1 G_2(t,s)\,ds = \frac{1}{k^2} \text{ for } t \in [0,1], \\ \sup_{t\in[0,1]} \int_0^1 G_2(t,s)\,ds + \sup_{t\in[0,1]} \int_0^1 \left|\frac{\partial G_2}{\partial t}(t,s)\right| ds \\ \quad = m_2 \leq \frac{1}{k^2} + \frac{1}{2}, \end{cases}$$

$$(2.13)' \qquad \overline{d_0} G_2(t,s) - \left|\frac{\partial G_2}{\partial t}(t,s)\right| \geq G_2(s,s) + \left|\frac{\partial G_2}{\partial t}(s-0,s)\right|$$

for $t,s \in [0,1]$ *and*

$$(2.14)' \qquad \overline{d_0} G_2(t,s) - \left|\frac{\partial G_2}{\partial t}(t,s)\right| \geq G_2(s,s) + \left|\frac{\partial G_2}{\partial t}(s+0,s)\right|,$$

where $s,t \in [0,1]$ *and* $\overline{d_0} \geq 2k \tan k/2 + 1$,

$$(2.15)' \qquad G_2(s,s) + \left|\frac{\partial G_2}{\partial t}(s-0,s)\right| \geq \overline{M_0} \left(G_2(t,s) + \left|\frac{\partial G_2}{\partial t}(t,s)\right|\right)$$

for $s,t \in [0,1]$, $\overline{M_0} \in \left(0, \frac{\cos k/2}{1+k\sin k/2}\right]$ *and*

$$(2.16)' \qquad G_2(s,s) + \left|\frac{\partial G_2}{\partial t}(s+0,s)\right| \geq \overline{M_0} \left(G_2(t,s) + \left|\frac{\partial G_2}{\partial t}(t,s)\right|\right),$$

where $s,t \in [0,1]$.

Throughout the paper

$\mathcal{D} = (-\infty, \infty) \times [0, \infty) \times (-\infty, \infty)$, $\tilde{\mathcal{D}} = (-\infty, \infty) \times (-\infty, 0] \times (-\infty, \infty)$,
$\mu > 0$, $a : (-\infty, \infty) \to (0, \infty)$ is continuous and 1-periodic, $L > 0$,

$$\phi_i(t) = \mu L \int_0^1 G_i(t,s)ds \quad \text{for } i = 1, 2,\ t \in [0, 1],$$

$\overline{\phi_i} : (-\infty, \infty) \to (-\infty, \infty)$, $\overline{\phi_i} \in P_1^2(\mathbb{R})$, $\overline{\phi_i}(t) = \phi_i(t)$ *for* $i = 1, 2$ *and* $t \in [0, 1]$,

$$m_i = \sup_{t\in[0,1]} \int_0^1 G_i(t,s)\, ds + \sup_{t\in[0,1]} \int_0^1 \left|\frac{\partial G_i}{\partial t}(t,s)\right| ds$$

for $i = 1, 2$.

3 Positive periodic solutions

In this section we present results on existence of positive 1-periodic solutions of the equations (1.1) and (1.2).

Theorem 3.1 *Suppose that*

(3.1) $$\begin{cases} f : \mathcal{D} \to (-\infty, \infty) \text{ is continuous,} \\ f(t+1, v_0, v_1) = f(t, v_0, v_1) \text{ for all } (t, v_0, v_1) \in \mathcal{D}, \\ \text{there exists a constant } L > 0 \text{ with} \\ f(t, v_0, v_1) + L \geq 0 \text{ for all } (t, v_0, v_1) \in \mathcal{D}, \end{cases}$$

(3.2) *there exists a function* $\psi(u)$ *such that* $f(t, v_0, v_1) + L \leq \psi(v_0 + |v_1|)$ *on* $\mathcal{D}$, *where* $\psi : [0, \infty) \to [0, \infty)$ *is continuous and nondecreasing and* $\psi(u) > 0$ *for* $u > 0$,

(3.3) $$\begin{cases} \text{there exist } C_1 > 0 \text{ and } r > 0 \text{ such that } r \geq \mu L C_1 d_0, \\ \int_0^1 G_1(t,s)ds \leq M_0 C_1 \text{ for } t \in [0, 1] \text{ and } \frac{r}{\psi(r + \|\phi_1\|_1)} \geq \mu m_1, \end{cases}$$

where d_0, M_0 *and* m_1 *have properties (2.9)–(2.11),*

(3.4) $$\begin{cases} f(t, v_0, v_1) + L \geq \tau(t) g(v_0) \text{ on } \mathcal{D}, \text{ where } \tau : (-\infty, \infty) \to [0, \infty) \\ \text{is continuous and 1-periodic and } g : [0, \infty) \to [0, \infty) \text{ is continuous,} \\ g(u) > 0 \text{ for } u > 0 \text{ and } g \text{ is nondecreacing,} \end{cases}$$

(3.5) *there exists* $R > 0$ *such that* $R > r$ *and*

$$d_0 R \leq \mu \int_0^1 \tau(s) \left[d_0 G_1\left(\frac{1}{2}, s\right) - \left|\frac{\partial G_1}{\partial t}\left(\frac{1}{2}, s\right)\right| \right] g\left(\frac{\varepsilon M_0 R}{d_0}\right) ds,$$

where $\varepsilon > 0$ *is any constant such that*

$$1 - \frac{\mu L C_1 d_0}{R} \geq \varepsilon.$$

Then (1.1) has a positive solution $x \in P_1^2(\mathbb{R})$.

Proof The proof of Theorem is similar to that of Theorem 2.1 in the paper [1]. To show (1.1) has a positive 1-periodic solution we will look at

$$x(t) = \mu \int_0^1 G_1(t,s) f_+^*(s, x(s) - \overline{\phi_1}(s), x'(s) - \overline{\phi_1'}(s))\, ds, \tag{3.6}$$

where

$$f_+^*(t, v_0, v_1) = \begin{cases} f(t, v_0, v_1) + L, & \text{if } (t, v_0, v_1) \in \mathcal{D} \\ f(t, 0, v_1) + L, & \text{if } (t, v_0, v_1) \in \tilde{\mathcal{D}}. \end{cases}$$

We will show that there exists a solution x_1 to (3.6) with $x_1(t) \geq \overline{\phi_1}(t)$ for $t \in [0,1]$. If this is true then $u(t) = x_1(t) - \overline{\phi_1}(t)$ is a positive solution of (3.6) since for $t \in [0,1]$ we have

$$\begin{aligned} u(t) &= \mu \int_0^1 G_1(t,s)[f_+^*(s, x_1(s) - \overline{\phi_1}(s), x_1'(s) - \overline{\phi_1'}(s))\, ds - \mu L \int_0^1 G_1(t,s)\, ds \\ &= \mu \int_0^1 G_1(t,s) f(s, u(s), u'(s))\, ds. \end{aligned}$$

We concentrate our study on (3.6). Let $E = (P_1^1(\mathbb{R}), \|\cdot\|_1)$ and

$$K_1 = \{u \in P_1^1(\mathbb{R}) : \min_{t\in[0,1]} [d_0 u(t) - |u'(t)| \geq M_0 \|u\|_1\}.$$

Obviously K_1 is a cone of E. Let

$$\Omega_1 = \{u \in P_1^1(\mathbb{R}) : \; \|u\|_1 < r\} \tag{3.7}$$

and

$$\Omega_2 = \{u \in P_1^1(\mathbb{R}) : \; \|u\|_1 < R\}. \tag{3.8}$$

Now let $A_1 : \; K_1 \cap (\overline{\Omega_2} \setminus \Omega_1) \to P_1^1(\mathbb{R})$ be defined by

$$A_1 \varphi = x_\varphi, \quad \text{where } \varphi \in K_1 \cap (\overline{\Omega_2} \setminus \Omega_1)$$

and x_φ is the unique 1-periodic solution of the equation

$$x''(t) - a^2(t) x(t) + \mu f_+^*(t, \varphi(t) - \overline{\phi_1}(t), \varphi'(t) - \overline{\phi_1}'(t)) = 0. \tag{3.9}$$

First we show $A_1 : \; K_1 \cap (\overline{\Omega_2} \setminus \Omega_1) \to K_1$. If $\varphi \in K_1 \cap (\overline{\Omega_2} \setminus \Omega_1)$ and $t \in [0,1]$, then by Lemma 2.4 we have

$$(A_1 \varphi)(t) = \mu \int_0^1 G_1(t,s) f_+^*(s, \varphi(s) - \overline{\phi_1}(s), \varphi'(s) - \overline{\phi_1}'(s))\, ds. \tag{3.10}$$

The relations (2.8)–(2.11) imply

$$\begin{cases} d_0(A_1\varphi)(t) - |(A_1\varphi)'(t)| = \\ \quad = \mu d_0 \int_0^1 G_1(t,s) f_+^*(s, \varphi(s) - \overline{\phi_1}(s), \varphi'(s) - \overline{\phi_1}'(s))\, ds \\ \quad - \mu \Big| \Big(\int_0^1 G_1(t,s) f_+^*(s, \varphi(s) - \overline{\phi}_1(s), \varphi'(s) - \overline{\phi_1}'(s))\, ds \Big)' \Big| \\ \quad \geq \mu \int_0^t \left[d_0 G_1(t,s) - \left| \frac{\partial G_1}{\partial t}(t,s) \right| \right] f_+^*(s, \varphi(s) - \overline{\phi_1}(s), \varphi'(s) - \overline{\phi_1}'(s))\, ds \\ \quad + \mu \int_t^1 \left[d_0 G_1(t,s) - \left| \frac{\partial G_1}{\partial t}(t,s) \right| \right] f_+^*(s, \varphi(s) - \overline{\phi_1}(s), \varphi'(s) - \overline{\phi_1}'(s))\, ds \\ \quad \geq \mu \int_0^t \left(G_1(s,s) + \left| \frac{\partial G_1}{\partial t}(s+0,s) \right| \right) f_+^*(s, \varphi(s) - \overline{\phi_1}(s), \varphi'(s) - \overline{\phi_1}'(s))\, ds \\ \quad + \mu \int_t^1 \left(G_1(s,s) + \left| \frac{\partial G_1}{\partial t}(s-0,s) \right| \right) f_+^*(s, \varphi(s) - \overline{\phi_1}(s), \varphi'(s) - \overline{\phi_1}'(s))\, ds \\ \quad \geq \mu M_0 \int_0^t \left(G_1(\overline{t},s) + \left| \frac{\partial G_1}{\partial t}(\overline{t},s) \right| \right) f_+^*(s, \varphi(s) - \overline{\phi_1}(s), \varphi'(s) - \overline{\phi_1}'(s))\, ds \\ \quad + \int_t^1 \left(G_1(\overline{t},s) + \left| \frac{\partial G_1}{\partial t}(\overline{t},s) \right| \right) f_+^*(s, \varphi(s) - \overline{\phi_1}(s), \varphi'(s) - \overline{\phi_1}'(s))\, ds \\ \quad \geq \mu M_0 \left(\int_0^1 \left(G_1(\overline{t},s) + \left| \frac{\partial G_1}{\partial t}(\overline{t},s) \right| \right) \right) f_+^*(s, \varphi(s) - \overline{\phi_1}(s), \varphi'(s) - \overline{\phi_1}'(s))\, ds \\ \quad \geq M_0 \left((A_1\varphi)(\overline{t}) + |(A_1\varphi)'(\overline{t})| \right), \quad \text{where } \overline{t} \in [0,1]. \end{cases}$$

Hence

$$d_0(A_1\varphi)(t) \geq d_0(A_1\varphi)(\overline{t}) - |(A_1\varphi)'(\overline{t})| \geq M_0 \|A_1\varphi\|_1. \tag{3.11}$$

Consequently $A_1\varphi \in K_1$. So $A_1 : \ K_1 \cap (\overline{\Omega_2} \setminus \Omega_1) \to K_1$. We now show

$$\|A_1\varphi\|_1 \leq \|\varphi\|_1 \qquad \text{for } \ \varphi \in K_1 \cap \partial\Omega_1. \tag{3.12}$$

To see this let $\varphi \in K_1 \cap \partial\Omega_1$. Then

$$\|\varphi\|_1 = r \quad \text{and} \quad \varphi(t) \geq \frac{M_0 r}{d_0} \quad \text{for } t \in \mathbb{R}.$$

From (3.2)–(3.3) we have

$$(A_1\varphi)(t) + |(A_1\varphi)'(t)| \leq \mu\psi(r + \|\overline{\phi_1}\|_1) m_1 \leq r \leq \|\varphi\|_1.$$

So (3.12) holds. Next we show

$$\|A_1\varphi\|_1 \geq \|\varphi\|_1 \quad \text{for } \varphi \in K_1 \cap \partial\Omega_2. \tag{3.13}$$

To see it let $\varphi \in K_1 \cap \partial\Omega_2$. Then $\|\varphi\|_1 = R$ and $d_0\varphi(t) \geq RM_0$ for $t \in \mathbb{R}$. Let ε be as in (3.5). From (3.3) we have

$$\begin{aligned} \varphi(t) - \overline{\phi_1}(t) = \varphi(t) - \mu L \int_0^1 G_1(t,s) ds \geq \varphi(t) - \frac{\mu L C_1 M_0 R d_0}{d_0 R} \\ \geq \varphi(t) \left(1 - \frac{\mu L C_1 d_0}{R} \right) \geq \varepsilon\varphi(t) \geq \frac{\varepsilon R M_0}{d_0} > 0. \end{aligned}$$

This together with (3.4)–(3.5) yields

$$d_0\|A_1\varphi\|_1 \geq d_0(A_1\varphi)\left(\frac{1}{2}\right) - \left|(A\varphi)'\left(\frac{1}{2}\right)\right|$$
$$\geq \mu\int_0^1 \left(d_0 G_1\left(\frac{1}{2},s\right) - \left|\frac{\partial G_1}{\partial t}\left(\frac{1}{2},s\right)\right|\right)\tau(s)g(\varphi(s)-\overline{\phi_1}(s))\,ds$$
$$\geq \mu\int_0^1 \tau(s)\left(d_0 G_1\left(\frac{1}{2},s\right) - \left|\frac{\partial G_1}{\partial t}\left(\frac{1}{2},s\right)\right|\right)g\left(\frac{\varepsilon M_0 R}{d_0}\right)ds \geq d_0 R.$$

Hence we have (3.13). We will show that A_1 is continuous and compact. To see it let

$$G_1(t,s) = \begin{cases} a_1(s)y_1(t) + a_2(s)y_2(t), & 0 \leq t \leq s \leq 1 \\ b_1(s)y_1(t) + b_2(s)y_2(t), & 0 \leq s \leq t \leq 1 \end{cases}$$

where (y_1, y_2) is a fundamental system of equation (2.1) and $a_i, b_i : [0,1] \to \mathbb{R}$ are continuous for $i = 1,2$. From relations (3.1)–(3.3) and properties of the function G_1 it follows that A_1 is a bounded and continuous operator. Notice that for $y \in K_1 \cap (\overline{\Omega_2} \setminus \Omega_1)$; $t_1, t_2 \in [0,1]$ and $t_1 < t_2$ that

$$|A_1 y)(t_2) - (A_1 y)(t_1)| \leq \int_0^1 |G_1(t_2,s) - G_1(t_1,s)|\psi(R + \|\overline{\phi_1}\|_1)\,ds$$

and

$$|(A_1y)'(t_2) - (A_1y)'(t_1)| \leq$$
$$\leq \int_0^{t_1} |b_1(s)(y_1'(t_2) - y_1'(t_1)) + b_2(s)(y_2'(t_2) - y_2'(t_1))|\psi(R + \|\overline{\phi_1}\|_1)\,ds$$
$$+ \int_{t_1}^{t_2} |b_1(s)y_1'(t_2) - a_1(s)y_1'(t_1) + b_2(s)y_2'(t_2) - a_2(s)y_2'(t_1)|\psi(R + \|\overline{\phi_1}\|_1)\,ds$$
$$+ \int_{t_2}^{1} |a_1(s)(y_1'(t_2) - y_1'(t_1)) + a_2(s)(y_2'(t_2) - y_2'(t_1))|\psi(R + \|\phi_1\|_1)\,ds$$
$$\leq \int_0^1 (|y_1'(t_2) - y_1'(t_1)| + |y_2'(t_2) - y_2'(t_1)|)h(s)\psi(R + \|\overline{\phi_1}\|_1)\,ds$$
$$+ 2\int_{t_1}^{t_2} (\|y_1\|_1 + \|y_2\|_1)h(s)\psi(R + \|\overline{\phi_1}\|_1)\,ds,$$

where $h(s) = |a_1(s)| + |a_2(s)| + |b_1(s)| + |b_2(s)|$.

Using the Arzela–Ascoli theorem we conclude that $A_1 : K_1 \cap (\overline{\Omega_2} \setminus \Omega_1) \to K_1$ is compact. Theorem 1.1 implies A_1 has a fixed point $x \in K_1 \cap (\overline{\Omega_2} \setminus \Omega_1)$, i.e. $r \leq \|x\|_1 \leq R$ and $x(t) \geq \frac{M_0 r}{d_0}$ for $t \in \mathbb{R}$. This completes the proof of Theorem 3.1. □

Theorem 3.2 *Suppose that*

(3.14) $f : \mathcal{D} \to [0,\infty)$ *is continuous*

(3.15) $f(t+1, v_0, v_1) = f(t, v_0, v_1)$ *for all* $(t, v_0, v_1) \in \mathcal{D}$,

$$(3.16)\quad \begin{cases} \text{there exist a function } \psi(u) \text{ such that} \\ f(t, v_0, v_1) \le \psi(v_0 + |v_1|) \text{ on } \mathcal{D}, \\ \text{where } \psi : [0,\infty) \to [0,\infty) \text{ is continuous and nondecreasing and} \\ \psi(u) > 0 \text{ for } u > 0, \end{cases}$$

(3.17) *there exists* r *such that* $r \ge \psi(r)\mu m_1$,

$$(3.18)\quad \begin{cases} \text{there exist function } \tau \text{ and } g \text{ such that } f(t, v_0, v_1) \ge \tau(t) g(v_0) \\ \text{for all } (t, v_0, v_1) \in \mathcal{D}, \text{ where } g : [0,\infty) \to [0,\infty),\ g(u) > 0 \\ \text{for } u > 0, g \text{ is continuous and nondecreasing and} \\ \tau : (-\infty,\infty) \to [0,\infty) \text{ is continuous and } 1\text{-periodic,} \end{cases}$$

(3.19) *there exists* $R > 0$ *such that* $R > r$ *and*

$$d_0 R \le \mu \int_0^1 \tau(s) \left[d_0 G_1\left(\frac{1}{2}, s\right) - \left| \frac{\partial G_1}{\partial t}\left(\frac{1}{2}, s\right) \right| \right] g\left(\frac{M_0 R}{d_0}\right) ds.$$

Then (1.1) has a positive solution $x \in P_1^2(\mathbb{R})$.

Proof The proof of Theorem 3.2 is similar to that of Theorem 3.1. Let E, Ω_1, Ω_2 and K_1 be as in Theorem 3.1. Now let $\varphi \in K_1 \cap (\overline{\Omega_2} \setminus \Omega_1)$ and let x_φ be the unique 1-periodic solution of the equation (3.9) and let $A_2 : K_1 \cap (\overline{\Omega_2} \setminus \Omega_1) \to P_1^1(\mathbb{R})$ be defined by $A_2\varphi = x_\varphi$. It is easy to check that $A_2 : K_1 \cap (\overline{\Omega_2} \setminus \Omega_1) \to K_1$, A_2 is continuous and compact, $\|A_2\varphi\|_1 \le \|\varphi\|_1$ for $\varphi \in K_1 \cap \partial\Omega_1$ and $\|A_2\varphi\| \ge \|\varphi\|_1$ for $\varphi \in K_1 \cap \partial\Omega_2$. Applying Theorem 1.1 we can show that the equation (1.1) has a positive solution $x \in P_1^2(\mathbb{R})$ which implies our assertion. □

Example 3.3 To illustrate the applicabillity of Theorem 3.2 we consider the following equation

$$x''(t) - x(t) + \mu(x(t) + |x'(t)|)^2 = 0. \tag{3.20}$$

Fix

$$a(t) \equiv 1, \quad \tau(t) = 1, \quad d_0 = \frac{3e-1}{2\sqrt{e}}, \quad M_0 = \frac{1}{\sqrt{e}}, \quad g(u) = \psi(u) = u^2,$$

We claim that (3.17) holds for $r \le \frac{2}{3\mu}$. To see this notice that $\mu m_1 \le \frac{3}{2}\mu$. Clearly

$$g\left(\frac{RM_0}{d_0}\right) = \frac{RM_0^2}{d_0^2} = \frac{4R^2}{(3e-1)^2}$$

and

$$\mu \int_0^1 \tau(s) \left[d_0 G_1 \left(\frac{1}{2}, s \right) - \left| \frac{\partial G_1}{\partial t} \left(\frac{1}{2}, s \right) \right| \right] g \left(\frac{M_0 R}{d_0} \right) ds$$
$$= \frac{4\mu R^2}{(3e-1)^2} \int_0^1 \left[\frac{(3e-1)}{2\sqrt{e}} G_1 \left(\frac{1}{2}, s \right) - \left| \frac{\partial G_1}{\partial t} \left(\frac{1}{2}, s \right) \right| \right] ds \ge \frac{(3e-1)}{2\sqrt{e}} R$$

for sufficiently large R. Thus all conditions of Theorem 3.2 are satisfied and the equation (3.20) has a positive solution $x \in P_1^2(\mathbb{R})$.

It is not difficult to verify that $x(t) = \frac{1}{\mu}$ is a periodic and positive solution of the equation (3.20).

Theorem 3.4 *Assume conditions (3.1)–(3.2) and (3.4). Suppose that*

(3.21) $\quad 0 < a(t) < \pi$ *for* $t \in [0, 1]$,

$$\text{(3.22)} \quad \begin{cases} \text{there exists } C_2 > 0 \text{ and } r > 0 \text{ such that } r \ge \mu L C_2 \overline{d}_0, \\ \int_0^1 G_2(t,s)\, ds \le C_2 \overline{M}_0 \text{ for } t \in [0,1] \text{ and } r \ge \psi(r + \|\overline{\phi}_2\|_1)\mu m_2, \\ \text{where } \overline{d_0} \text{ and } \overline{M}_0 \text{ have properties (2.13)–(2.16),} \end{cases}$$

$$\text{(3.23)} \quad \begin{cases} \text{there exists } R > 0 \text{ such that } R > r \text{ and} \\ \overline{d}_0 R \le \mu \int_0^1 \tau(s) \left[\overline{d}_0 G_2 \left(\frac{1}{2}, s \right) - \left| \frac{\partial G_2}{\partial t} \left(\frac{1}{2}, s \right) \right| \right] g \left(\frac{\varepsilon \overline{M}_0 R}{\overline{d}_0} \right) ds, \\ \text{where } \varepsilon > 0 \text{ is any constant such that } 1 - \frac{\mu L C_2 \overline{d}_0}{R} \ge \varepsilon. \end{cases}$$

Then (1.2) has a positive solution $x \in P_1^2(\mathbb{R})$.

Proof Let E, Ω_1 and Ω_2 be as in Theorem 3.1. Let

$$K_2 = \{u \in P_1^1(\mathbb{R}) : \min_{t \in [0,1]} [\overline{d}_0 u(t) - |u'(t)|] \ge \overline{M}_0 \|u\|_1\}.$$

Then K_2 is a cone of E. Now let $\varphi \in K_2 \cap (\overline{\Omega_2} \setminus \Omega_1)$ and let x_φ be the unique 1-periodic solution of the equation

$$x''(t) + a^2(t) x(t) = \mu f_+^*(t, \varphi(t) - \overline{\phi}_2(t), \varphi'(t) - \overline{\phi}_2'(t)),$$

where f_+^* is defined by (3.6). Finally let $A_3 : K_2 \cap (\overline{\Omega_2} \setminus \Omega_1) \to P_1^1(\mathbb{R})$ be defined by $A_3 \varphi = x_\varphi$. It is not difficult to prove that $A_3 : K_2 \cap (\overline{\Omega_2} \setminus \Omega_1) \to K_2$, A_3 is continuous and compact. The similar arguments as in Theorem 3.1 gurantee that $\|A_3 \varphi\|_1 \le \|\varphi\|_1$ for $\varphi \in K_2 \cap \partial\Omega_1$ and $\|A_3 \varphi\|_1 \ge \|\varphi\|_1$ for $\varphi \in K_2 \cap \partial\Omega_2$. Theorems 1.1 implies that A_3 has a fixed point $x \in K_2 \cap (\overline{\Omega_2} \setminus \Omega_1)$ i.e. $x(t) \ge \frac{\overline{M}_0 r}{\overline{d}_0}$ for $t \in \mathbb{R}$. This completes the proof of Theorem 3.4. □

In a similar way we can prove

Corollary 3.5 *Assume conditions (3.14)–(3.16) and (3.18). Suppose that*

(3.24) *there exists* $r > 0$ *such that* $r \geq \psi(r)\mu m_2$,

(3.25) *there exists* $R > 0$ *such that* $R > r$ *and*

$$\overline{d}_0 R \leq \mu \int_0^1 \tau(t) \left[\overline{d}_0 G_2\left(\frac{1}{2}, s\right) - \left|\frac{\partial G_2}{\partial t}\left(\frac{1}{2}, s\right)\right|\right] g\left(\frac{M_0 R}{\overline{d}_0}\right) ds.$$

Then (1.2) *has a positive solution* $x \in P_1^2(\mathbb{R})$.

Example 3.6 We consider the equation

(3.26) $$x''(t) + x(t) = \mu|\sin \pi t|[(x(t) + |x'(t)|)^2 - 1].$$

It is not difficult to verify that the equation (3.26) for $0 < \mu \leq 1/5$ has a solution x such that $x(t) > 0$ for $t \in \mathbb{R}$ and $x \in P_1^2(\mathbb{R})$. To see this we apply Theorem 3.4 with $a(t) \equiv 1$, $L = 1$, $\tau(t) = |\sin \pi t|$, $\overline{d}_0 = 2(\tan\frac{1}{2} + 1)$, $\overline{M}_0 = \frac{\cos 1/2}{1 + \sin 1/2}$, $g(u) = \psi(u) = u^2$, $\overline{\phi}_2 = \mu$, $C_2 = 2$, $r = 1$ and with sufficiently large R ($R > 1$).

4 Negative periodic solutions

In a similar way we can prove theorems on existence of negative periodic solutions of the equations (1.1) and (1.2).

Theorem 4.1 *Suppose that*

(4.1) $$\begin{cases} f : \tilde{\mathcal{D}} \to (-\infty, \infty) \text{ is continuous,} \\ f(t+1, v_0, v_1) = f(t, v_0, v_1) \text{ for } (t, v_0, v_1) \in \tilde{\mathcal{D}}, \\ \text{there exists a constant } L > 0 \text{ with} \\ f(t, v_0, v_1) - L \leq 0 \text{ for } (t, v_0, v_1) \in \tilde{\mathcal{D}}, \end{cases}$$

(4.2) $$\begin{cases} \text{there exists a function } \psi(u) \text{ such that} \\ -f(t, v_0, v_1) + L \leq \psi(|v_0| + |v_1|) \text{ for } (t, v_0, v_1) \in \tilde{\mathcal{D}}, \\ \text{where } \psi : [0, \infty) \to [0, \infty) \text{ is continuous} \\ \text{and nondecreasing and } \psi(u) > 0 \text{ for } u > 0, \end{cases}$$

(4.3) $L - f(t, v_0, v_1) \geq \tau(t) g(|v_0|)$ *for* $(t, v_0, v_1) \in \tilde{\mathcal{D}}$, *where* τ *and* g *have property (3.4),*

(4.4) *there exist* $R > 0$ *and* $r > 0$ *such that (3.3) and (3.5) hold.*

Then (1.1) has a negative solution $x \in P_1^2(\mathbb{R})$.

Proof Let

$$f_-^*(t,v_0,v_1)=\begin{cases} f(t,v_0,v_1)-L, & \text{if } (t,v_0,v_1)\in\tilde{\mathcal{D}}\\ f(t,0,v_1)-L, & \text{if } (t,v_0,v_1)\in\mathcal{D}. \end{cases}$$

We will show that there exists a solution x_2 to the following equation

$$x(t)=\mu\int_0^1 G_1(t,s)f_-^*(s,x(s)+\overline{\phi}_1(s),x'(s)+\overline{\phi}_1'(s))\,ds \tag{4.5}$$

with $x_2(t)+\overline{\phi}_1(t)<0$ for $t\in[0,1]$. If this is true, then $u(t)=x_2(t)+\overline{\phi}_1(t)$ is a negative solution of the equation (1.1) since for $t\in[0,1]$ we have

$$u(t)=\mu\int_0^1 G_1(t,s)f(s,u(s),u'(s))\,ds.$$

Let Ω_1,Ω_2 and E be as in Theorem 3.1. Now let

$$K_3=\{u\in P_1^1(\mathbb{R}):\ \max_{t\in[0,1]}[d_0u(t)+|u'(t)|]\le -M_0\|u\|_1\}.$$

Then K_3 is a cone of E. Let $\varphi\in K_3\cap(\overline{\Omega_2}\setminus\Omega_1)$ and let x_φ be the unique 1-periodic solution of the equation

$$x''(t)-a^2(t)x(t)+\mu f_-^*(t,\varphi(t)+\overline{\phi}_1(t),\varphi'(t)+\overline{\phi}_1'(t))=0.$$

Finally let $A_4: K_3\cap(\overline{\Omega_2}\setminus\Omega_1)\to P_1^1(\mathbb{R})$ be defined by $A_4\varphi=x_\varphi$. Then

$$(A_4\varphi)(t)=\mu\int_0^1 G_1(t,s)f_-^*(s,\varphi(s)+\overline{\phi}_1(s),\varphi'(s)+\overline{\phi}_1'(s))\,ds$$

for $t\in[0,1]$. By Lemma 2.6 we have.

$$\left\{\begin{aligned}
&d_0(A_4\varphi)(t)+|(A_4\varphi)'(t)|\\
&\quad\le\mu\textstyle\int_0^1\left[d_0G_1(t,s)-\left|\frac{\partial G_1}{\partial t}(t,s)\right|\right]f_-^*(s,\varphi(s)+\overline{\phi}_1(s),\varphi'(s)+\overline{\phi}_1'(s))\,ds\\
&\quad=\mu\textstyle\int_0^t\left[d_0G_1(t,s)-\left|\frac{\partial G_1}{\partial t}(t,s)\right|\right]f_-^*(s,\varphi(s)+\phi_1(s),\varphi'(s)+\overline{\phi_1}'(s))\,ds\\
&\quad\ +\mu\textstyle\int_t^1\left[d_0G_1(t,s)-\left|\frac{\partial G_1}{\partial t}(t,s)\right|\right]f_-^*(s,\varphi(s)+\phi_1(s),\varphi'(s)+\overline{\phi_1}'(s))\,ds\\
&\quad\le\mu\textstyle\int_0^t\left[G_1(s,s)+\left|\frac{\partial G_1}{\partial t}(s+0,s)\right|\right]f_-^*(s,\varphi(s)+\overline{\phi_1}(s),\varphi'(s)+\overline{\phi_1}'(s))\,ds\\
&\quad\ +\mu\textstyle\int_t^1\left[G_1(s,s)+\left|\frac{\partial G_1}{\partial t}(s-0,s)\right|\right]f_-^*(s,\varphi(s)+\overline{\phi}_1(s),\varphi'(s)+\overline{\phi}_1'(s))\,ds.
\end{aligned}\right.$$

Hence, by (2.10)–(2.11) we get

$$\begin{gathered}d_0(A_4\varphi)(t)+|(A_4\varphi)'(t)|\\ \le-\mu M_0\int_0^1\left[G_1(\overline{t},s)+\left|\frac{\partial G_1}{\partial t}(\overline{t},s)\right|\right](-f_-^*(s,\varphi(s)+\overline{\phi}_1(s),\varphi'(s)+\overline{\phi}_1'(s)))\,ds,\end{gathered}$$

where $\overline{t}\in[0,1]$. So

$$d_0(A_4\varphi)(t)+|(A_4\varphi)'(t)|\le -M_0\|A_4\varphi\|_1.$$

Consequently $A_4: K_3 \cap (\overline{\Omega_2} \setminus \Omega_1) \to K_3$. Using arguments similar to those in the proof of Theorem 3.1 we conclude that A_4 is continuous and compact. Let $\varphi \in K_3 \cap \partial\Omega_1$. Then $\|A_4\varphi\|_1 \le \|\varphi\|_1$. If $\varphi \in K_3 \cap \partial\Omega_2$, then $\|\varphi\|_1 = R$ and $d_0\varphi(t) \le -RM_0$.

Now let ε be as in (3.5). Then by (3.3) we have

$$\begin{cases} \varphi(t) \le \varphi(t) + \overline{\phi}_1(t) \le \varphi(t) + \mu L \int_0^1 G_1(t,s)\,ds \le \varphi(t) + \mu L M_0 C_1 \\ \quad \le -\frac{RM_0}{d_0} + \frac{\mu L M_0 C_1 R d_0}{d_0 R} = -\frac{RM_0}{d_0}\left(1 - \frac{\mu L C_1 d_0}{R}\right) \le -\frac{\varepsilon R M_0}{d_0} < 0. \end{cases}$$

(for $t \in [0,1]$). This together with (3.5) and (4.3) yields

$$\begin{cases} -d_0\|A_4\varphi\|_1 \le d_0(A_4\varphi)\left(\frac{1}{2}\right) + \left|(A_4\varphi)'\left(\frac{1}{2}\right)\right| \\ \le \mu \int_0^1 \left[d_0 G_1\left(\frac{1}{2}, s\right) - \left|\frac{\partial G_1}{\partial t}\left(\frac{1}{2}, s\right)\right|\right] \left[f(s, \varphi(s) + \overline{\phi}_1(s), \varphi'(s) + \overline{\phi}_1'(s)) - L\right] ds \\ \le -\mu \int_0^1 \left[d_0 G_1\left(\frac{1}{2}, s\right) - \left|\frac{\partial G_1}{\partial t}\left(\frac{1}{2}, s\right)\right|\right] \tau(s) g(|\varphi(s) + \overline{\phi}_1(s)|)\,ds \\ \le -\mu \int_0^1 \left[d_0 G_1\left(\frac{1}{2}, s\right) - \left|\frac{\partial G_1}{\partial t}\left(\frac{1}{2}, s\right)\right|\right] \tau(s) g\left(\frac{\varepsilon R M_0}{d_0}\right) ds \le -d_0 R. \end{cases}$$

So $\|A_4\varphi\|_1 \ge R = \|\varphi\|_1$. By Theorem 1.1 the operator A_4 has at least one fixed point in the set $K_3 \cap (\overline{\Omega_2} \setminus \Omega_1)$, which means that (1.1) has a negative solution x such that $x \in P_1^2(\mathbb{R})$. This completes the proof of Theorem 4.1. □

By the same way we can prove the following

Corollary 4.2 *Suppose that*

(4.6) $f : \tilde{\mathcal{D}} \to (-\infty, 0]$ *is continuous*

(4.7) $f(t+1, v_0, v_1) = f(t, v_0, v_1)$ *for all* $(t, v_0, v_1) \in \tilde{\mathcal{D}}$,

(4.8) *there exists a function* ψ *such that*

$$|f(t, v_0, v_1)| \le \psi(v_0 + |v_1|) \quad \text{on } \tilde{\mathcal{D}},$$

where $\psi : [0,\infty) \to [0,\infty)$ *is continuous and nondecreasing and* $\psi(u) > 0$ *for* $u > 0$,

(4.9) *there exist functions* τ *and* g *such that*

$$-f(t, v_0, v_1) \ge \tau(t) g(|v_0|) \quad \text{for } (t, v_0, v_1) \in \tilde{\mathcal{D}},$$

where τ *and* g *have property (3.4),*

(4.10) *there exist constants* r *and* R *having properties (3.17) and (3.19).*

Then (1.1) has a negative solution $x \in P_1^2(\mathbb{R})$.

Theorem 4.3 *Assume that conditions (4.1)–(4.3), (3.21)–(3.23) are satisfied. Then (1.2) has a negative solution* $x \in P_1^2(\mathbb{R})$.

Proof The proof of Theorem 4.3 is similar to that of Theorem 4.1. Let Ω_1, Ω_2, f_-^* and E be as in Theorem 4.1. Let

$$K_4 = \left\{ u \in P_1^1(\mathbb{R}) : \max_{t\in[0,1]} [\overline{d}_0 u(t) + |u'(t)|] \le -\overline{M}_0 \|u\|_1 \right\}.$$

Obviously K_4 is a cone of E. We will show there exists a solution x_3 of the equation

$$x(t) = \mu \int_0^1 G_2(t,s) f_-^*(s, x(s) + \overline{\phi}_2(s), x'(s) + \overline{\phi}_2'(s))\, ds$$

with $x_3(t) + \overline{\phi}_2(t) < 0$ for $t \in [0,1]$. Let $\varphi \in K_4 \cap (\overline{\Omega_2} \setminus \Omega_1)$ and let x_φ be the unique 1-periodic solution of the equation

$$x''(t) + a^2(t)x(t) = \mu f_-^*(t, \varphi(t) + \overline{\phi}_2(t), \varphi'(t) + \overline{\phi}_2'(t)).$$

Finally, let $A_5 : K_4 \cap (\overline{\Omega_2} \setminus \Omega_1) \to P_1^1(\mathbb{R})$ be defined by $A_5\varphi = x_\varphi$. Then

$$(A_5\varphi)(t) = \mu \int_0^1 G_2(t,s) f_-^*(s, \varphi(s) + \overline{\phi}_2(s), \varphi'(s) + \overline{\phi}_2'(s))\, ds$$

for $t \in [0,1]$. By Lemma 2.7 we have

$$\begin{aligned} &\overline{d}_0 (A_5\varphi)(t) + |(A_5\varphi)'(t)| \\ &\le -\mu\overline{M}_0 \int_0^1 \left[G_2(\overline{t}, s) + \left| \frac{\partial G_2}{\partial t}(\overline{t}, s) \right| \right] \left(-f_-^*(s, \varphi(s) + \overline{\phi}_2(s), \varphi'(s) + \overline{\phi}_2(s)) \right) ds, \end{aligned}$$

where $\overline{t} \in [0,1]$. So

$$\overline{d}_0 (A_5\varphi)(t) + |(A_5\varphi)'(t)| \le -\overline{M}_0 \|A_5\varphi\|_1.$$

Consequently $A_5 : K_4 \cap (\overline{\Omega_2} \setminus \Omega_1) \to K_4$. Also A_5 is continuous and compact. Let $\varphi \in K_4 \cap \partial\Omega_1$. Then $\|A_5\varphi\|_1 \le \|\varphi\|_1$. If $\varphi \in K_4 \cap \partial\Omega_2$, then $\overline{d}_0 \varphi(t) \le -R\overline{M}_0$ and

$$\varphi(t) \le \varphi(t) + \overline{\phi}_2(t) \le \frac{-R\overline{M}_0}{\overline{d}_0} \left(\frac{1 - \mu L C_2 \overline{d}_0}{R} \right) \le \frac{-\varepsilon R\overline{M}_0}{\overline{d}_0} < 0,$$

where ε is as in (3.22). This together with (4.3) yields

$$\begin{aligned} -\overline{d}_0 \|A_5\varphi\|_1 &\le \overline{d}_0 (A_5\varphi)\left(\frac{1}{2}\right) + \left| (A_5\varphi)'\left(\frac{1}{2}\right) \right| \\ &\le -\mu \int_0^1 \left[\overline{d}_0 G_2\left(\frac{1}{2}, s\right) - \left| \frac{\partial G_2}{\partial t}\left(\frac{1}{2}, s\right) \right| \right] [f(s, \varphi(s) + \overline{\phi}_2(s), \varphi'(s) + \overline{\phi}_2(s)) - L]\, ds \\ &\le -\mu \int_0^1 \left[\overline{d}_0 G_2\left(\frac{1}{2}, s\right) - \left| \frac{\partial G_2}{\partial t}\left(\frac{1}{2}, s\right) \right| \right] \tau(s) g\left(\frac{\varepsilon R\overline{M}_0}{\overline{d}_0} \right) ds \le -\overline{d}_0 R. \end{aligned}$$

Thus $\|A_5\varphi\|_1 \ge \|\varphi\|_1$. By Theorem 1.1 the operator A_5 has at least on fixed point in the set $K_4 \cap (\overline{\Omega_2} \setminus \Omega_1)$ which means that (1.2) has a negative solution x such that $x \in P_1^2(\mathbb{R})$. This completes the proof of Theorem 4.3. □

In the simlar way we can prove the following

Corollary 4.4 *Assume that conditions (4.6)–(4.9), (3.24)–(3.25) are satisfied. Then (1.2) has a negative solution* $x \in P_1^2(\mathbb{R})$.

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On Applications of the Yano–Ako Operator*

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Abstract

In this paper we consider a method by which a skew-symmetric tensor field of type (1,2) in M_n can be extended to the tensor bundle $T_q^0(M_n)$ $(q > 0)$ on the *pure cross-section.* The results obtained are to some extend similar to results previously established for cotangent bundles $T_1^0(M_n)$. However, there are various important differences and it appears that the problem of lifting tensor fields of type (1,2) to the tensor bundle $T_q^0(M_n)$ $(q > 1)$ on the *pure cross-section* presents difficulties which are not encountered in the case of the cotangent bundle.

Key words: Lift; tensor bundle; pure tensor; operator Yano–Ako.

2000 Mathematics Subject Classification: 53C15, 53C25, 53C55

1 Introduction

Let M_n be a differentiable manifold of class C^∞ and finite dimension n, and let $T_q^0(M_n)$ $(q > 0)$ be the bundle over M_n of tensors of type $(0, q)$:

$$T_q^0(M_n) = \bigcup_{P \in M_n} T_q^0(P),$$

where $T_q^0(P)$ denotes the tensor spaces of tensors of type $(0, q)$ at $P \in M_n$.

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i. $\pi : T_q^0(M_n) \to M_n$ is the projection $T_q^0(M_n)$ onto M_n.

ii. The indices $i, j, \dots$ run from 1 to n, the indices $\bar{\imath}, \bar{\jmath}, \dots$ from $n+1$ to $n + n^q = \dim T_q^0(M_n)$ and the indices $I = (i, \bar{\imath}), J = (j, \bar{\jmath}), \dots$ from 1 to $n + n^q$. The so-called Einsteins summation convention is used.

iii. $\Im(M)$ is the ring of real-valued C^∞ functions on M_n. $T_q^p(M_n)$ is the module over $\Im(M)$ of C^∞ tensor fields of type (p, q).

iv. Vector fields in M_n are denoted by $V, W, \dots$ The Lie derivation with respect to V is denoted by L_V.

Denoting by x^j the local coordinates of $P = \pi(\tilde{P})$ ($\tilde{P} \in T_q^0(M_n)$) in a neighborhood $U \subset M_n$ and if we make $(x^j, t_{j_1 \dots j_q}) = (x^j, x^{\bar{\jmath}})$ correspond to the point $\tilde{P} \in \pi^{-1}(U)$, we can introduce a system of local coordinates $(x^j, x^{\bar{\jmath}})$ in a neighborhood $\pi^{-1}(U) \subset T_q^0(M_n)$, where $t_{j_1 \dots j_q} \overset{\text{def}}{=} x^{\bar{\jmath}}$ are components of $t \in T_q^0(P)$ with respect to the natural frame ∂_i.

If $\alpha \in T_q^0(M_n)$, it is regarded, in a natural way (by contraction), as a function in $T_q^0(M_n)$, which we denote by $i\alpha$. If α has the local expression $\alpha = \alpha^{j_1 \cdots j_q} \partial_{j_1} \otimes \dots \otimes \partial_{j_q}$ in a coordinate neighborhood $U(x^i) \subset M_n$, then $i\alpha$ has the local expression $i\alpha = \alpha(t) = \alpha^{j_1 \cdots j_q} t_{j_1 \dots j_q}$ with respect to the coordinates $(x^j, x^{\bar{\jmath}})$ in $\pi^{-1}(U)$.

Suppose that $A \in T_q^0(M_n)$. We define the vertical lift ${}^VA \in T_0^1(T_q^0(M_n))$ of A to $T_q^0(M_n)$ (see [1]) by ${}^VA(i\alpha) = \alpha(A) \circ \pi = {}^V(\alpha(A))$, where ${}^V(\alpha(A))$ is the vertical lift of the function $\alpha(A) \in \Im(M_n)$. The vertical lift ${}^VA$ of A to $T_q^0(M_n)$ has components

$$ {}^VA = \begin{pmatrix} {}^VA^j \\ VA^{\bar{\jmath}} \end{pmatrix} = \begin{pmatrix} 0 \\ A_{j_1 \dots j_q} \end{pmatrix} \tag{1.1} $$

with respect to the coordinates $(x^j, x^{\bar{\jmath}})$ in $T_q^0(M_n)$.

We define the complete lift ${}^CV = \bar{L}_V$ of V to $T_q^0(M_n)$ (see [1]) by ${}^CV(i\alpha) = i(L_V\alpha)$, $\alpha \in T_0^q(M_n)$. The complete lift ${}^CV$ of V to $T_q^0(M_n)$ has components

$$ {}^CV^k = V^k, \qquad {}^CV^{\bar{k}} = -\sum_{\lambda=1}^{q} t_{k_1 \dots s \dots k_q} \partial_{k_\lambda} V^s \tag{1.2} $$

with respect to the coordinates $(x^k, x^{\bar{k}})$ in $T_q^0(M_n)$.

Suppose that there is given a tensor field $\xi \in T_q^0(M_n)$. Then the correspondence $x \to \xi_x$, ξ_x being the value of ξ at $x \in M_n$, determines a mapping $\sigma_\xi : M_n \to T_q^0(M_n)$ such that $\pi \circ \sigma_\xi = id_{M_n}$, and the n dimensional submanifold $\sigma_\xi(M_n)$ of $T_q^0(M_n)$ is called the cross-section determined by ξ. If the tensor field ξ has the local components $\xi_{k_1 \dots k_q}(x^k)$, the cross-section $\sigma_\xi(M_n)$ is locally expressed by $x^k = x^k$, $x^{\bar{k}} = \xi_{k_1 \dots k_q}(x^k)$ with respect to the coordinates $(x^k, x^{\bar{k}})$ in $T_q^0(M_n)$. Differentiating by x^j, we see that the n tangent vector fields B_j to $\sigma_\xi(M_n)$have components

$$ (B_j^K) = \left(\frac{\partial x^K}{\partial x^j} \right) = \begin{pmatrix} \delta_j^k \\ \partial_j \xi_{k_1 \dots k_q} \end{pmatrix} \tag{1.3} $$

with respect to the natural frame $\{\partial_k, \partial_{\bar{k}}\}$ in $T^0_q(M_n)$.

On the other hand, the fibre is locally expressed by x^k = const, $t_{k_1\ldots k_q} = t_{k_1\ldots k_q}$, $t_{k_1\ldots k_q}$ being consider as parameters. Thus, on differentiating with respect to $x^{\bar{j}} = t_{j_1\ldots j_q}$, we see that the n^q tangent vector fields $C_{\bar{j}}$ to the fibre have components

$$(C^K_{\bar{j}}) = \left(\frac{\partial x^K}{\partial x^{\bar{j}}}\right) = \begin{pmatrix} 0 \\ \delta^{j_1}_{k_1}\ldots\delta^{j_q}_{k_q} \end{pmatrix} \tag{1.4}$$

with respect to the natural frame $\{\partial_k, \partial_{\bar{k}}\}$ in $T^0_q(M_n)$.

We consider in $\pi^{-1}(U) \subset T^0_q(M_n)$, $n+n^q$ local vector fields B_j and $C_{\bar{j}}$ along $\sigma_\xi(M_n)$. They form a local family of frames $\{B_j, C_{\bar{j}}\}$ along $\sigma_\xi(M_n)$, which is called the adapted (B,C)-frame of $\sigma_\xi(M_n)$ in $\pi^{-1}(U)$. Taking account of (1.2), we can easily prove that , the complete lift ${}^CV$ has along $\sigma_\xi(M_n)$ components of the form

$${}^CV = \begin{pmatrix} {}^C\tilde{V}^j \\ {}^C\tilde{V}^{\bar{j}} \end{pmatrix} = \begin{pmatrix} V^j \\ -(L_V\xi)_{j_1\ldots j_q} \end{pmatrix} \tag{1.5}$$

with respect to the adapted (B,C)-frame [2], where $(L_V\xi)_{j_1\ldots j_q}$ are local components of $L_V\xi$ in M_n.

2 The vertical-vector lift of a tensor field of type (1,1)

Let $\varphi \in T^1_1(M_n)$. Making use of the Jacobian matrix of the coordinate transformation in $T^0_q(M_n)$:

$$x^{i'} = x^{i'}(x^i), x^{\bar{i}'} = t_{(i')} = A^{(i)}_{(i')}t_{(i)}$$
$$= A^{(i)}_{(i')}x^{\bar{i}}\big(t_{(i)} = t_{i_1\ldots i_q}, A^{(i)}_{(i')} = A^{i_1}_{i'_1}\ldots A^{i_q}_{i'_q}, A^i_{i'} = \frac{\partial x_i}{\partial x_{i'}}\big)$$

we can define a vector field $\gamma\varphi \in T^1_0(T^0_q(M_n))$ [3]:

$$\gamma\varphi = ((\gamma\varphi)^J) = \begin{pmatrix} 0 \\ t_{ji_2\ldots i_q}\varphi^j_{i_1} \end{pmatrix},$$

where $\varphi^j_{i_1}$ are local components of φ in M_n. Clearly, we have $(\gamma\varphi)({}^Vf) = 0$ for any $f \in \Im(M_n)$, so that $\gamma\varphi$ is a vertical vector field. We call $\gamma\varphi$ the vertical-vector lift of the tensor field $\varphi \in T^1_1(M_n)$ to $T^0_q(M_n)$. We can easily verify that the vertical-vector lift $\gamma\varphi$ has along $\sigma_\xi(M_n)$ components

$$\gamma\varphi = ((\gamma\tilde{\varphi})^I) = \begin{pmatrix} 0 \\ \xi_{ji_2\ldots i_q}\varphi^j_{i_1} \end{pmatrix}$$

with respect to the adapted (B,C)-frame, where $\xi_{i_1\ldots i_q}$ are local components of ξ in M_n.

Let S be an element of $T^1_2(M_n)$ with local components S^k_{ij} in M_n. In a similar way, if $\gamma((L_{V_1}S)_{V_2})$, $\gamma((L_{V_2}S)_{V_1})$ and $\gamma(S_{[V_1,V_2]})$ are vertical-vector lifts of $(L_{V_1}S)_{V_2} = (v_2^m (L_{V_1}S)^j_{im}) \in T^1_1(M_n)$, $(L_{V_2}S)_{V_1} = (v_1^m (L_{V_2}S)^j_{im}) \in T^1_1(M_n)$ and $S_{[V_1,V_2]} = (S^j_{im}[V_1,V_2]^m) \in T^1_1(M_n)$, respectively, then $\gamma((L_{V_1}S)_{V_2})$, $\gamma((L_{V_2}S)_{V_1})$ and $\gamma(S_{[V_1,V_2]})$ have along $\sigma_\xi(M_n)$ respectively components of the form

$$\gamma((L_{V_1}S)_{V_2}) = (\gamma((\tilde{L}_{V_1}S)_{V_2})^I) = \begin{pmatrix} 0 \\ \xi_{ji_2\ldots i_q} v_2^m (L_{V_1}S)^j_{i_1 m} \end{pmatrix},$$

$$\gamma((L_{V_2}S)_{V_1}) = (\gamma((\tilde{L}_{V_2}S)_{V_1})^I) = \begin{pmatrix} 0 \\ \xi_{ji_2\ldots i_q} v_1^m (L_{V_2}S)^j_{i_1 m} \end{pmatrix},$$

$$\gamma(S_{[V_1,V_2]}) = (\gamma(\tilde{S}_{[V_1,V_2]})^I) = \begin{pmatrix} 0 \\ \xi_{ji_2\ldots i_q} S^j_{i_1 m}[V_1,V_2]^m \end{pmatrix}$$

with respect to the adapted (B,C)-frame, where $[V_1,V_2] = L_{V_1}V_2$.

3 The complete lift of a skew-symmetric tensor field of type (1,2)

Suppose now that $S \in T^1_2(M_n)$ is a skew-symmetric tensor field of type (1,2) with local components S^k_{ij}, that is $S(V,W) = -S(W,V)$, $\forall V,W \in T^1_0(M_n)$. A tensor field $\xi \in T^0_q(M_n)$ is called pure with respect to $S \in T^1_2(M_n)$, if [4]:

$$\begin{cases} S^r_{k_1 j_1}\xi_{r\ldots j_q} = \ldots = S^r_{k_1 j_q}\xi_{j_1\ldots r}, \\ S^r_{j_1 k_2}\xi_{r\ldots j_q} = \ldots = S^r_{j_q k_2}\xi_{j_1\ldots r}. \end{cases}$$

In particular, covector fields will be considered to be pure. Let $\overset{*}{T}{}^0_q(M_n)$denotes a module of all the tensor fields $\xi \in T^0_q(M_n)$ which are pure with respect to S. We consider a pure cross-section $\sigma^S_\xi(M_n)$ determined by $\xi \in \overset{*}{T}{}^0_q(M_n)$. We observe that the local vector fields

$$^C X_{(i)} = {}^C\left(\frac{\partial}{\partial x^i}\right) = {}^C\left(\delta^h_i \frac{\partial}{\partial x^h}\right) = \begin{pmatrix} \delta^h_i \\ 0 \end{pmatrix}$$

and

$$^V X^{(\bar{\imath})} = {}^V(dx^{i_1} \otimes \ldots \otimes dx^{i_q}) = {}^V(\delta^{i_1}_{h_1} \ldots \delta^{i_q}_{h_q} dx^{h_1} \otimes \ldots \otimes dx^{h_q}) = \begin{pmatrix} 0 \\ \delta^{i_1}_{h_1} \ldots . \delta^{i_q}_{h_q} \end{pmatrix}$$

$$i = 1, \ldots, n,\ \bar{\imath} = n+1, \ldots, n+n^q$$

span the module of vector fields in $\pi^{-1}(U) \subset T^0_q(M_n)$. Hence any tensor field is determined in $\pi^{-1}(U)$ by its action of $^C X_{(i)}$ and $^V X^{(\bar{\imath})}$. Then we define a tensor field ${}^CS \in T_2^1(T_q^0(M_n))$ along the pure cross-section $\sigma_\xi^S(M_n)$ by

$$\begin{cases} {}^CS({}^CV_1, {}^CV_2) = {}^C(S(V_1,V_2)) - \gamma((L_{V_2}S)_{V_1}) \\ \quad + \gamma((L_{V_1}S)_{V_2}) + \gamma(S_{[V_1,V_2]}), \quad \forall V_1, V_2 \in T_0^1(M_n) & \text{(i)} \\ {}^CS({}^VA, {}^CV_2) = {}^V(S_{V_2}(A)), \quad \forall A \in T_q^1(M_n), & \text{(ii)} \\ {}^CS({}^CV_1, {}^VB) = {}^V(S_{V_1}(B)), \quad \forall B \in T_q^1(M_n), & \text{(iii)} \\ {}^CS({}^VA, {}^VB) = 0, & \text{(iv)} \end{cases} \tag{3.1}$$

where $S_{V_2}(A), S_{V_1}(B) \in T_q^0(M_n)$ and call ${}^CS$ the complete lift of $S \in T_2^1(M_n)$ to $T_q^0(M_n)$ along $\sigma_\xi^S(M_n)$.

Let ${}^C\tilde{S}^J_{L_1L_2}$ be components of ${}^CS$ with respect to the adapted (B,C)-frame of the pure cross-section $\sigma_\xi^S(M_n)$. From (1.1), (1.3), (1.4) and ${}^VA = {}^V\tilde{A}^jB_j + {}^V\tilde{A}^{\bar{j}}C_{\bar{j}}$, we easily obtain ${}^V\tilde{A}^j = 0$, ${}^V\tilde{A}^{\bar{j}} = {}^VA^{\bar{j}} = A_{j_1\ldots j_q}$. Thus the vertical lift ${}^VA$ also has components of the form (1.1) with respect to the adapted (B,C)-frame of $\sigma_\xi^S(M_n)$. Then, from (3.1) we have

$$\begin{cases} {}^C\tilde{S}^J_{L_1L_2}\,{}^C\tilde{V}_1^{L_1}\,{}^C\tilde{V}_2^{L_2} = {}^C(\tilde{S}(V_1,V_2))^J - \gamma((\tilde{L}_{V_2}S)_{V_1})^J \\ \quad + \gamma((\tilde{L}_{V_1}S)_{V_2})^J + \gamma(\tilde{S}_{[V_1,V_2]})^J, & \text{(i)} \\ {}^C\tilde{S}^J_{L_1L_2}\,{}^V\tilde{A}^{L_1}\,{}^C\tilde{V}_2^{L_2} = {}^V(S_{V_2}(\tilde{A}))^J & \text{(ii)} \\ {}^C\tilde{S}^J_{L_1L_2}\,{}^C\tilde{V}_1^{L_1}\,{}^V\tilde{B}^{L_2} = {}^V(S_{V_1}(\tilde{B}))^J, & \text{(iii)} \\ {}^C\tilde{S}^J_{L_1L_2}\,{}^V\tilde{A}^{L_1}\,{}^V\tilde{B}^{L_2} = 0, & \text{(iv)} \end{cases} \tag{3.2}$$

where

$${}^V(S_{V_2}\tilde{(A)})^J = \begin{pmatrix} 0 \\ S^m_{j_1l}V_2^lA_{mj_2\ldots j_q} \end{pmatrix}, \qquad {}^V(S_{V_1}\tilde{(B)})^J = \begin{pmatrix} 0 \\ S^m_{lj_1}V_1^lB_{mj_2\ldots j_q} \end{pmatrix}.$$

When $J = j$, from (i) of (3.2) we have

$${}^C\tilde{S}^j_{l_1l_2} = S^j_{l_1l_2}, \quad {}^C\tilde{S}^j_{\bar{l}_1l_2} = {}^C\tilde{S}^j_{l_1\bar{l}_2} = {}^C\tilde{S}^j_{\bar{l}_1\bar{l}_2} = 0,$$

where $x^{\bar{l}_a} = t_{r_1\ldots r_q}$, $a = 1, 2$.

When $J = \bar{j}$, (i) of (3.2) reduces to

$$\begin{aligned} {}^C\tilde{S}^{\bar{j}}_{l_1l_2}\,{}^C\tilde{V}_1^{l_1}\,{}^C\tilde{V}_2^{l_2} &+ {}^C\tilde{S}^{\bar{j}}_{\bar{l}_1l_2}\,{}^C\tilde{V}_1^{\bar{l}_1}\,{}^C\tilde{V}_2^{l_2} + {}^C\tilde{S}^{\bar{j}}_{l_1\bar{l}_2}\,{}^C\tilde{V}_1^{l_1}\,{}^C\tilde{V}_2^{\bar{l}_2} \\ &+ {}^C\tilde{S}^{\bar{j}}_{\bar{l}_1\bar{l}_2}\,{}^C\tilde{V}_1^{\bar{l}_1}\,{}^C\tilde{V}_2^{\bar{l}_2} + \xi_{ij_2\ldots j_q}v_1^m(L_{V_2}S)^i_{j_1m} \\ - \xi_{ij_2\ldots j_q}v_2^m(L_{V_1}S)^i_{j_1m} &- \xi_{ij_2\ldots j_q}S^i_{j_1m}[V_1,V_2]^m = {}^C(\tilde{S}(V_1,V_2))^{\bar{j}} \end{aligned} \tag{3.3}$$

Now, using the Generalized Yano–Ako operator we will investigate components ${}^C\tilde{S}^{\bar{j}}_{l_1l_2}$. The Generalized Yano–Ako operator on the pure module $\overset{*}{T}{}_q^0(M_n)$ is given by [4], [5].

$$\begin{aligned} (\Phi_S\xi)_{l_1l_2j_1\ldots j_q} = S^m_{l_1l_2}\partial_m\xi_{j_1\ldots j_q} - \partial_{l_1}(S^m_{j_1l_2}\xi_{mj_2\ldots j_q}) - \partial_{l_2}(S^m_{l_1j_1}\xi_{mj_2\ldots j_q}) \\ + \sum_{a=1}^q(\partial_{j_a}S^m_{l_1l_2})\xi_{j_1\ldots m\ldots j_q}. \end{aligned}$$

After some calculations we have

$$\begin{aligned} &V_2^{l_2}V_1^{l_1}(\Phi_{S(V_1,V_2)}\xi)_{l_1l_2j_1\ldots j_q} + V_1^{l_1}S_{l_1j_1}^m L_{V_2}\xi_{mj_2\ldots j_q} + V_2^{l_2}S_{j_1l_2}^m L_{V_1}\xi_{mj_2\ldots j_q} \\ &+ V_2^{l_2}(L_{V_1}S_{j_1l_2}^m)\xi_{mj_2\ldots j_q} - V_1^{l_1}(L_{V_2}S_{j_1l_1}^m)\xi_{mj_2\ldots j_q} + (L_{V_1}V_2)^{l_1}S_{j_1l_1}^m)\xi_{mj_2\ldots j_q} \\ &\qquad = L_{S(V_1,V_2)}\xi_{j_1\ldots j_q} \end{aligned} \tag{3.4}$$

for any $V_1, V_2 \in T_0^1(M_n)$. Using (1.5), from (3.4) we have

$$\begin{aligned} &((\Phi_{S(V_1,V_2)}\xi)_{l_1l_2j_1\ldots j_q})^C\tilde{V}_1^{l_1\,C}\tilde{V}_2^{l_2} - S_{l_1j_1}^{r_1}\delta_{j_2}^{r_2}\ldots\delta_{j_q}^{r_q\,C}\tilde{V}_1^{l_1\,C}\tilde{V}_2^{\bar{l}_2} \\ &- S_{j_1l_2}^{r_1}\delta_{j_2}^{r_2}\ldots\delta_{j_q}^{r_q\,C}\tilde{V}_1^{\bar{l}_1\,C}\tilde{V}_2^{l_2} + V_2^{l_2}(L_{V_1}S_{j_1l_2}^m)\xi_{mj_2\ldots j_q} - V_1^{l_1}(L_{V_2}S_{j_1l_1}^m)\xi_{mj_2\ldots j_q} \\ &+ (L_{V_1}V_2)^{l_1}S_{j_1l_1}^m\xi_{mj_2\ldots j_q} = -^C(\tilde{S}(V_1,V_2))^{\bar{j}}. \end{aligned} \tag{3.5}$$

Comparing (3.3) and (3.5), we get

$$^C\tilde{S}_{l_1l_2}^{\bar{j}} = -(\Phi_S\xi)_{l_1l_2j_1\ldots j_q}.$$

By similar devices, from (ii)–(iv) of (3.2) we have also

$$^C\tilde{S}_{\bar{l}_1\bar{l}_2}^{\bar{j}} = 0, \quad ^C\tilde{S}_{\bar{l}_1l_2}^{\bar{j}} = S_{j_1l_2}^{r_1}\delta_{j_2}^{r_2}\ldots\delta_{j_q}^{r_q}, \quad ^C\tilde{S}_{l_1\bar{l}_2}^{\bar{j}} = S_{l_1j_1}^{r_1}\delta_{j_2}^{r_2}\ldots\delta_{j_q}^{r_q}.$$

Thus the complete lift ^CS of $S \in T_2^1(M_n)$ $(S(V,W) = -S(W,V))$ has along the pure cross-section $\sigma_\xi^S(M_n)$ components

$$\begin{cases} ^C\tilde{S}_{l_1l_2}^{j} = S_{l_1l_2}^{j}, \ ^C\tilde{S}_{\bar{l}_1l_2}^{j} = {}^C\tilde{S}_{l_1\bar{l}_2}^{j} = {}^C\tilde{S}_{\bar{l}_1\bar{l}_2}^{j} = {}^C\tilde{S}_{\bar{l}_1\bar{l}_2}^{\bar{j}} = 0 \\ ^C\tilde{S}_{\bar{l}_1l_2}^{\bar{j}} = S_{j_1l_2}^{r_1}\delta_{j_2}^{r_2}\ldots\delta_{j_q}^{r_q}, \ ^C\tilde{S}_{l_1\bar{l}_2}^{\bar{j}} = S_{l_1j_1}^{r_1}\delta_{j_2}^{r_2}\ldots\delta_{j_q}^{r_q}, \\ ^C\tilde{S}_{l_1l_2}^{\bar{j}} = -(\Phi_S\xi)_{l_1l_2j_1\ldots j_q} \end{cases} \tag{3.6}$$

with respect to the adapted (B,C)-frame of $\sigma_\xi^S(M_n)$, where $\Phi_S\xi$ is the Generalized Yano–Ako operator.

Remark 1 ^CS in the form (3.6) is unique solution of (3.1). Therefore, if $\overset{*}{S}$ is element of $T_2^1(T_q^0(M_n))$, such that

$$\begin{cases} ^C\overset{*}{S}(^CV_1,{}^CV_2) = {}^C(S(V_1,V_2)) - \gamma((L_{V_2}S)_{V_1}) \\ \quad + \gamma((L_{V_1}S)_{V_2}) + \gamma(S_{[V_1,V_2]}), \\ ^C\overset{*}{S}(^VA,{}^CV_2) = {}^V(S_{V_2}(A)), \\ ^C\overset{*}{S}(^CV_1,{}^VB) = {}^V(S_{V_1}(B)), \\ ^C\overset{*}{S}(^VA,{}^VB) = 0, \end{cases}$$

then $\overset{*}{S} = {}^CS$.

Remark 2 The equation (3.1) is a useful extension of the equation $^CV(i\alpha) = i(L_V\alpha)$, $\alpha \in T_0^q(M_n)$ (see §1) to tensor fields of type (1,2) along the pure cross-section $\sigma_\xi^S(M_n)$.

In the case $\partial_m \xi_{j_1 \ldots j_q} = 0$, (B, C)-frame is considered as a natural frame $\{\partial_h, \partial_{\bar{h}}\}$ of $\sigma_\xi^S(M_n)$. Then, from (3.6) we obtain components of ${}^C S$ along the pure cross-section with respect to the natural frame $\{\partial_h, \partial_{\bar{h}}\}$ of $\sigma_\xi^S(M_n)$ in $\pi^{-1}(U)$ (see [5]). The diagonal and horizontal lifts for tensor fields of special kinds to the tensor bundle have been studied in [6]–[8].

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Density of a Family of Linear Varietes*

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Abstract

The measurability of the family, made up of the family of plane pairs and the family of lines in 3-dimensional space A_3, is stated and its density is given.

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1 Introduction

A measure on a family of geometric objects can be introduced by assigning to each object a point of an auxiliary space and considering a suitable measure on that space. In general the dimension of the auxiliary space is equal to the number of parameters on which the geometric objects depend. A basic problem is to specify measures which are invariant with respect to a given group of transformations which map the family onto itself.

This problem was first considered by Crofton [3] who specified the invariant measure on the family of all straight lines in Euclidean 2-space E^2. This was extended to E^3 by Deltheil [4] and Chern [1] first considered families of geometric objects in projective space.

Santaló [9] calculated measures of certain families of varieties with respect to three different groups and found that these were equal. Stoka [10] studied the family of parabolas. He proved that a family is measurable if it is measurable with respect to its maximal group of invariance

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However Cirlincione [2] found a measurable family of varieties even though the family was not measurable with respect to the maximal group of invariance. This proves that the Stoka's condition is not necessary.

In Section 2 we provide background and definitions and in Section 3 we prove that the family of varieties, where each variety is a pair consisting of two hyperplanes and a straight line in 3-dimensional affine space A_3 is measurable.

2 Background

Let $\mathcal{H}_n$ be an n-dimensional space with coordinates $x_1, x_2, \ldots, x_n$ in which a Lie group of transformations acts.

Let G_r be one of its subgroups defined by the equations

$$y_i = f_i(x_1, x_2, \ldots, x_n; a_1, a_2, \ldots, a_r) \qquad (i = 1, 2, \ldots, n) \qquad (\#)$$

where $a_1, a_2, \ldots, a_r$ are basic parameters.

Definition 1 The function $F(x_1, x_2, \ldots, x_n)$ is an integral invariant function of the group (#), if

$$\int_{\mathcal{A}_x} F(x_1, x_2, \ldots, x_n)\, dx_1 dx_2 \ldots dx_n = \int_{\mathcal{A}_y} F(y_1 y_2, \ldots, y_n)\, dy_1 dy_2 \ldots dy_n)$$

for each measurable set of points $\mathcal{A}_x$ of the space $\mathcal{H}_n$.

Theorem 1 *The integral invariant functions of the group (#) are the solutions of the following Deltheil's system of partial differential equations:*

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left[\xi_h^i(x) F(x)\right] = 0 \qquad (h = 1, 2, \ldots, r),$$

where $\xi_h^i(x)$ are the coefficients of the infinitesimal transformations of the group (#) (see [4], p. 28).

Definition 2 A measurable Lie group of transformations is a group which admits only one integral invariant function (up to a multiplicative constant).

Let G be a group which leaves globally invariant a family $\Im$ of varietes in $\mathcal{H}_n$. To G there is associated a group H (isomorphic to G) of transformations acting on the (auxiliary) space of parameters of the family.

Definition 3 A family $\Im$ is measurable with respect to G if H is measurable in the sense of Definition 2. If Φ is its integral invariant function, then the measure of $\Im$ with respect to the group G is given by

$$\mu_G = \int_{\mathcal{A}_\alpha} \Phi(\alpha_1, \alpha_2, \ldots, \alpha_q)\, d\alpha_1 d\alpha_2 \ldots d\alpha_q,$$

where $\mathcal{A}_\alpha$ is the set of points of the auxiliary space which corresponds to the family $\Im$.

Definition 4 A family $\Im$ of varieties is measurable if the measures with respect to every group of invariance of the family are equal, if they exist.

Theorem 2 (Stoka's first condition) *If the group $\overline{H}$ associated to the maximal group of invariance of $\Im$ (where the only transformation, which leaves invariant each element of the family, is the identity) is measurable, the family is measurable.*

Theorem 3 (Stoka's second condition) *If $\overline{H}$ is not measurable and there are two measurable subgroups with different integral invariant functions, then $\Im$ is not measurable.*

3 Measurability of the family $\Im_{10}$

Theorem 4 *The family of varieties,where each variety is consisted of two planes and a straight line in 3-dimensional affine space A_3, is measurable.*

Let us consider the family of plane pairs and the family of lines in the affine space A_3 (suppose that planes and lines are in general position)

$$\Im_{10}: \begin{cases} b_1x_1 + b_2x_2 + b_3x_3 = 1, \\ c_1x_1 + c_2x_2 + c_3x_3 = 1, \\ x_1 = l_1x_3 + q_1 \\ x_2 = l_2x_3 + q_2 \end{cases}$$

which depend on 10 parameters $b_1, b_2, b_3, c_1, c_2, c_3, l_1, l_2, q_1, q_2$.

Let G_{12} be the affinity group given by the equations

$$G_{12}: \begin{cases} x_1 = p_{11}x_1' + p_{12}x_2' + p_{13}x_3' + \alpha_1 \\ x_2 = p_{21}x_1' + p_{22}x_2' + p_{23}x_3' + \alpha_2 \\ x_3 = p_{31}x_1' + p_{32}x_2' + p_{33}x_3' + \alpha_3 \end{cases}$$

and let $\sum_{i=1}^{3} b_i\alpha_i \neq 1$, $\sum_{i=1}^{3} c_i\alpha_i \neq 1$.

We put

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \quad C = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix},$$

$$L = \begin{pmatrix} l_1 \\ l_2 \\ 1 \end{pmatrix}, \quad Q = \begin{pmatrix} q_1 \\ q_2 \\ 0 \end{pmatrix}, \quad X' = \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix},$$

$$P = (p_{ij}) \;\; (i, j = 1, 2, 3) \text{ with } \det P \neq 0, \quad A = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$$

so that we obtain

$$\begin{aligned} &\Im_{10} \,:\, {}^tB\cdot X = 1,\ {}^tC\cdot X = 1,\ X = L\cdot x_3 + Q \\ &G_{12} \,:\, X = P\cdot X' + A \end{aligned} \tag{1}$$

Now we see how the 10 parameters of the family $\Im_{10}$ change by applying any transformation T of G_{12}.

With a similar meaning of B', C', L', Q' the new variety is given by the equations

$${}^tB'\cdot X' = 1, {}^tC'\cdot X' = 1 \tag{2}$$

$$X' = L'\cdot x_3' + Q'. \tag{3}$$

From (1) we have

$$\begin{aligned} {}^tB\cdot X &= {}^tB\cdot(P\cdot X' + A) = {}^tB\cdot P\cdot X' + {}^tB\cdot A = 1 \\ {}^tC\cdot X &= {}^tC\cdot(P\cdot X' + A) = {}^tC\cdot P\cdot X' + {}^tC\cdot A = 1 \end{aligned}$$

hence

$${}^tB\cdot P\cdot X' = 1 - {}^tB\cdot A, \qquad {}^tC\cdot P\cdot X' = 1 - {}^tC\cdot A.$$

Finally, dividing by $1-\,{}^tB\cdot A$ and $1-\,{}^tC\cdot A$ respectively, we obtain

$$\frac{1}{1-{}^tB\cdot A}({}^tB\cdot P)X' = 1, \qquad \frac{1}{1-{}^tC\cdot A}({}^tC\cdot P)X' = 1. \tag{4}$$

In the same way we obtain

$$\begin{aligned} X &= L\cdot x_3 + Q \Rightarrow P\cdot X' + A = L\cdot(p_{31}x_1' + p_{32}x_2' + p_{33}x_3' + \alpha_3) + Q \\ &\Rightarrow P\cdot X' = L\cdot(p_{31}x_1' + p_{32}x_2' + p_{33}x_3' + \alpha_3) + Q - A \end{aligned}$$

i.e.

$$\begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix} \cdot \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} l_1 \\ l_2 \\ 1 \end{pmatrix} \cdot (p_{31}x_1' + p_{32}x_2' + p_{33}x_3' + \alpha_3) + \begin{pmatrix} q_1 - \alpha_1 \\ q_2 - \alpha_2 \\ 0 - \alpha_3 \end{pmatrix}$$

or equivalently

$$\begin{aligned} p_{11}x_1' + p_{12}x_2' + p_{13}x_3' &= l_1(p_{31}x_1' + p_{32}x_2' + p_{33}x_3') + l_1\alpha_3 + (q_1 - \alpha_1) \\ p_{21}x_1' + p_{22}x_2' + p_{23}x_3' &= l_2(p_{31}x_1' + p_{32}x_2' + p_{33}x_3') + l_2\alpha_3 + (q_2 - \alpha_2) \\ p_{33}x_3' &= p_{33}x_3' + \alpha_3 + (0 - \alpha_3) \end{aligned}$$

hence

$$\begin{aligned} (p_{11} - l_1p_{31})x_1' + (p_{12} - l_1p_{32})x_2' &= (l_1p_{33} - p_{13})x_3' + l_1\alpha_3 + (q_1 - \alpha_1) \\ (p_{21} - l_2p_{31})x_1' + (p_{22} - l_2p_{32})x_2' &= (l_2p_{33} - p_{23})x_3' + l_2\alpha_3 + (q_2 - \alpha_2) \\ p_{33}x_3' &= p_{33}x_3' \end{aligned}$$

Omitting the last identity and using the matrix form, we have

$$\begin{pmatrix} p_{11}-l_1p_{31} & p_{12}-l_1p_{32} \\ p_{21}-l_2p_{31} & p_{22}-l_2p_{32} \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} l_1p_{33}-p_{13} \\ l_2p_{33}-p_{23} \end{pmatrix} x_3' + \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} \alpha_3 + \begin{pmatrix} q_1-\alpha_1 \\ q_2-\alpha_2 \end{pmatrix} \tag{5}$$

Putting

$$R = \begin{pmatrix} p_{11}-l_1p_{31} & p_{12}-l_1p_{32} \\ p_{21}-l_2p_{31} & p_{22}-l_2p_{32} \end{pmatrix},$$

we have

$$R^{-1} = \frac{1}{\triangle} \begin{pmatrix} p_{22}-l_2p_{32} & -p_{12}+l_1p_{32} \\ -p_{21}+l_2p_{31} & p_{22}-l_1p_{31} \end{pmatrix}$$

where

$$\triangle = \|R\| = (p_{11}-l_1p_{31})(p_{22}-l_2p_{32}) - (p_{12}-l_2p_{32})(p_{21}-l_2p_{31}).$$

Then we can write (5) as

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = R^{-1} \begin{pmatrix} l_1p_{33}-p_{13} \\ l_2p_{33}-p_{23} \end{pmatrix} x_3' + R^{-1} \left[\begin{pmatrix} l_1 \\ l_2 \end{pmatrix} \alpha_3 + \begin{pmatrix} q_1-\alpha_1 \\ q_2-\alpha_2 \end{pmatrix} \right]$$

or

$$\begin{aligned} \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} &= \frac{1}{\triangle} \begin{pmatrix} p_{22}-l_2p_{32} & -p_{12}+l_1p_{32} \\ -p_{21}+l_2p_{31} & p_{11}-l_1p_{31} \end{pmatrix} \cdot \begin{pmatrix} l_1p_{33}-p_{13} \\ l_2p_{33}-p_{23} \end{pmatrix} x_3' \\ &+ \frac{1}{\triangle} \begin{pmatrix} p_{22}-l_2p_{32} & -p_{12}+l_1p_{32} \\ -p_{21}+l_2p_{31} & p_{11}-l_1p_{31} \end{pmatrix} \cdot \begin{pmatrix} l_1\alpha_3+q_1-\alpha_1 \\ l_2\alpha_3+q_2-\alpha_2 \end{pmatrix} \end{aligned} \tag{6}$$

By comparing (2) and (3) with (4) and (6) respectively, we have the connections between the new parameters b_1', b_2', b_3', c_1', c_2', c_3', l_1', l_2', q_1', q_2' and the initial ones:

$$\begin{aligned} b_1' &= \sum_{i=1}^{3} b_ip_{i1} \cdot \frac{1}{1-\sum_{i=1}^{3} b_i\alpha_i} & b_2' &= \sum_{i=1}^{3} b_ip_{i2} \cdot \frac{1}{1-\sum_{i=1}^{3} b_i\alpha_i} \\ b_3' &= \sum_{i=1}^{3} b_ip_{i3} \cdot \frac{1}{1-\sum_{i=1}^{3} b_i\alpha_i} & c_1' &= \sum_{i=1}^{3} c_ip_{i1} \cdot \frac{1}{1-\sum_{i=1}^{3} c_i\alpha_i} \\ c_2' &= \sum_{i=1}^{3} c_ip_{i2} \cdot \frac{1}{1-\sum_{i=1}^{3} c_i\alpha_i} & c_3' &= \sum_{i=1}^{3} c_ip_{i3} \cdot \frac{1}{1-\sum_{i=1}^{3} c_i\alpha_i} \\ l_1' &= \frac{1}{\triangle} [(p_{22}-l_2p_{32})(l_1p_{33}-p_{13}) + (-p_{12}+l_1p_{32})(l_2p_{33}-p_{23})] \\ l_2' &= \frac{1}{\triangle} [(-p_{21}+l_2p_{31})(l_1p_{33}-p_{13}) + (p_{11}-l_1p_{31})(l_2p_{33}-p_{23})] \\ q_1' &= \frac{1}{\triangle} [(p_{22}-l_2p_{32})(l_1\alpha_3+q_1-\alpha_1) + (-p_{12}+l_1p_{32})(l_2\alpha_3+q_2-\alpha_2)] \\ q_2' &= \frac{1}{\triangle} [(-p_{21}+l_2p_{31})(l_1\alpha_3+q_1-\alpha_1) + (p_{11}-l_1p_{31})(l_2\alpha_3+q_2-\alpha_2)] \end{aligned} \tag{7}$$

In the 10-dimensional parameter space $\mathcal{A}_{10}$, (7) are the equations of group H_{12} which is associated to G_{12} (operating in 3-dimensional space A_3).

Also group H_{12} depends on *twelve* parameters p_{11}, p_{21}, p_{31}, p_{12}, p_{22}, p_{32}, p_{13}, p_{23}, p_{33}, α_1, α_2, α_3 and the *unit* $o \in H_{12}$ (as for G_{12}) corresponds to the values of the parameters

$$p_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases} \quad \text{and} \quad \alpha_i = 0 \quad (i, j = 1, 2, 3).$$

Now we construct the matrix whose elements are the coefficients of the infinitesimal transfomations of H_{12} and we note that the columns of this matrix are the derivates of b'_1, b'_2, b'_3, c'_1, c'_2, c'_3, l'_1, l'_2, q'_1, q'_2 with respect to parameters p_{ij}, $i, j = 1, 2, 3$ and α_i, $i = 1, 2, 3$:

$$\begin{array}{lll}
\left(\frac{\partial b'_1}{\partial p_{11}}\right)_o = b_1, & \left(\frac{\partial b'_2}{\partial p_{11}}\right)_o = 0, & \left(\frac{\partial b'_3}{\partial p_{11}}\right)_o = 0, \\
\left(\frac{\partial b'_1}{\partial p_{21}}\right)_o = b_2, & \left(\frac{\partial b'_2}{\partial p_{21}}\right)_o = 0, & \left(\frac{\partial b'_3}{\partial p_{21}}\right)_o = 0, \\
\left(\frac{\partial b'_1}{\partial p_{31}}\right)_o = b_3, & \left(\frac{\partial b'_2}{\partial p_{31}}\right)_o = 0, & \left(\frac{\partial b'_3}{\partial p_{31}}\right)_o = 0, \\
\left(\frac{\partial b'_1}{\partial p_{12}}\right)_o = 0, & \left(\frac{\partial b'_2}{\partial p_{12}}\right)_o = b_1, & \left(\frac{\partial b'_3}{\partial p_{12}}\right)_o = 0, \\
\left(\frac{\partial b'_1}{\partial p_{22}}\right)_o = 0, & \left(\frac{\partial b'_2}{\partial p_{22}}\right)_o = b_2, & \left(\frac{\partial b'_3}{\partial p_{22}}\right)_o = 0, \\
\left(\frac{\partial b'_1}{\partial p_{32}}\right)_o = 0, & \left(\frac{\partial b'_2}{\partial p_{32}}\right)_o = b_3, & \left(\frac{\partial b'_3}{\partial p_{32}}\right)_o = 0, \\
\left(\frac{\partial b'_1}{\partial p_{13}}\right)_o = 0, & \left(\frac{\partial b'_2}{\partial p_{13}}\right)_o = 0, & \left(\frac{\partial b'_3}{\partial p_{13}}\right)_o = b_1, \\
\left(\frac{\partial b'_1}{\partial p_{23}}\right)_o = 0, & \left(\frac{\partial b'_2}{\partial p_{23}}\right)_o = 0, & \left(\frac{\partial b'_3}{\partial p_{23}}\right)_o = b_2, \\
\left(\frac{\partial b'_1}{\partial p_{33}}\right)_o = 0, & \left(\frac{\partial b'_2}{\partial p_{33}}\right)_o = 0, & \left(\frac{\partial b'_3}{\partial p_{33}}\right)_o = b_3, \\
\left(\frac{\partial b'_1}{\partial \alpha_1}\right)_o = b_1^2, & \left(\frac{\partial b'_2}{\partial \alpha_1}\right)_o = b_2 b_1, & \left(\frac{\partial b'_3}{\partial \alpha_1}\right)_o = b_3 b_1, \\
\left(\frac{\partial b'_1}{\partial \alpha_2}\right)_o = b_1 b_2, & \left(\frac{\partial b'_2}{\partial \alpha_2}\right)_o = b_2^2, & \left(\frac{\partial b'_3}{\partial \alpha_2}\right)_o = b_3 b_2, \\
\left(\frac{\partial b'_1}{\partial \alpha_3}\right)_o = b_1 b_3, & \left(\frac{\partial b'_2}{\partial \alpha_3}\right)_o = b_2 b_3, & \left(\frac{\partial b'_3}{\partial \alpha_3}\right)_o = b_3^2, \\
\left(\frac{\partial c'_1}{\partial p_{11}}\right)_o = c_1, & \left(\frac{\partial c'_2}{\partial p_{11}}\right)_o = 0, & \left(\frac{\partial c'_3}{\partial p_{11}}\right)_o = 0, \\
\left(\frac{\partial c'_1}{\partial p_{21}}\right)_o = c_2, & \left(\frac{\partial c'_2}{\partial p_{21}}\right)_o = 0, & \left(\frac{\partial c'_3}{\partial p_{21}}\right)_o = 0, \\
\left(\frac{\partial c'_1}{\partial p_{31}}\right)_o = c_3, & \left(\frac{\partial c'_2}{\partial p_{31}}\right)_o = 0, & \left(\frac{\partial c'_3}{\partial p_{31}}\right)_o = 0, \\
\left(\frac{\partial c'_1}{\partial p_{12}}\right)_o = 0, & \left(\frac{\partial c'_2}{\partial p_{12}}\right)_o = c_1, & \left(\frac{\partial c'_3}{\partial p_{12}}\right)_o = 0, \\
\left(\frac{\partial c'_1}{\partial p_{22}}\right)_o = 0, & \left(\frac{\partial c'_2}{\partial p_{22}}\right)_o = c_2, & \left(\frac{\partial c'_3}{\partial p_{22}}\right)_o = 0, \\
\left(\frac{\partial c'_1}{\partial p_{32}}\right)_o = 0, & \left(\frac{\partial c'_2}{\partial p_{32}}\right)_o = c_3, & \left(\frac{\partial c'_3}{\partial p_{32}}\right)_o = 0, \\
\left(\frac{\partial c'_1}{\partial p_{13}}\right)_o = 0, & \left(\frac{\partial c'_2}{\partial p_{13}}\right)_o = 0, & \left(\frac{\partial c'_3}{\partial p_{13}}\right)_o = c_1,
\end{array}$$

$$
\begin{array}{llll}
\left(\frac{\vartheta c'_1}{\partial p_{23}}\right)_o = 0, & \left(\frac{\partial c'_2}{\partial p_{23}}\right)_o = 0, & \left(\frac{\partial c'_3}{\partial p_{23}}\right)_o = c_2, & \\
\left(\frac{\partial c'_1}{\partial p_{33}}\right)_o = 0, & \left(\frac{\partial c'_2}{\partial p_{33}}\right)_o = 0, & \left(\frac{\partial c'_3}{\partial p_{33}}\right)_o = c_3, & \\
\left(\frac{\partial c'_1}{\partial \alpha_1}\right)_o = c_1^2, & \left(\frac{\partial c'_2}{\partial \alpha_1}\right)_o = c_2c_1, & \left(\frac{\partial c'_3}{\partial \alpha_1}\right)_o = c_3c_1, & \\
\left(\frac{c'_1}{\partial \alpha_2}\right)_o = c_1c_2, & \left(\frac{\partial c'_2}{\partial \alpha_2}\right)_o = c_2^2, & \left(\frac{\partial c'_3}{\partial \alpha_2}\right)_o = c_3c_2, & \\
\left(\frac{\partial c'_1}{\partial \alpha_3}\right)_o = c_1c_3, & \left(\frac{\partial c'_2}{\partial \alpha_3}\right)_o = c_2c_3, & \left(\frac{\partial c'_3}{\partial \alpha_3}\right)_o = c_3^2, & \\
\left(\frac{\partial l'_1}{\partial p_{11}}\right)_o = -l_1, & \left(\frac{\partial l'_2}{\partial p_{11}}\right)_o = 0, & \left(\frac{\partial q'_1}{\partial p_{11}}\right)_o = -q_1, & \left(\frac{\partial q'_2}{\partial p_{11}}\right)_o = 0, \\
\left(\frac{\partial l'_1}{\partial p_{21}}\right)_o = 0, & \left(\frac{\partial l'_2}{\partial p_{21}}\right)_o = -l_1, & \left(\frac{\partial q'_1}{\partial p_{21}}\right)_o = 0, & \left(\frac{\partial q'_2}{\partial p_{21}}\right)_o = -q_1, \\
\left(\frac{\partial l'_1}{\partial p_{31}}\right)_o = l_1^2, & \left(\frac{\partial l'_2}{\partial p_{31}}\right)_o = l_1l_2, & \left(\frac{\partial q'_1}{\partial p_{31}}\right)_o = l_1q_1, & \left(\frac{\partial q'_2}{\partial p_{31}}\right)_o = l_2q_1, \\
\left(\frac{\partial l'_1}{\partial p_{12}}\right)_o = -l_2, & \left(\frac{\partial l'_2}{\partial p_{12}}\right)_o = 0, & \left(\frac{\partial q'_1}{\partial p_{12}}\right)_o = -q_2, & \left(\frac{\partial q'_2}{\partial p_{12}}\right)_o = 0, \\
\left(\frac{\partial l'_1}{\partial p_{22}}\right)_o = 0, & \left(\frac{\partial l'_2}{\partial p_{22}}\right)_o = -l_2, & \left(\frac{\partial q'_1}{\partial p_{22}}\right)_o = 0, & \left(\frac{\partial q'_2}{\partial p_{22}}\right)_o = -q_2, \\
\left(\frac{\partial l'_1}{\partial p_{32}}\right)_o = l_1l_2, & \left(\frac{\partial l'_2}{\partial p_{32}}\right)_o = l_2^2, & \left(\frac{\partial q'_1}{\partial p_{32}}\right)_o = l_1q_2, & \left(\frac{\partial q'_2}{\partial p_{32}}\right)_o = l_2q_2, \\
\left(\frac{\partial l'_1}{\partial p_{13}}\right)_o = -1, & \left(\frac{\partial l'_2}{\partial p_{13}}\right)_o = 0, & \left(\frac{\partial q'_1}{\partial p_{13}}\right)_o = 0, & \left(\frac{\partial q'_2}{\partial p_{13}}\right)_o = 0, \\
\left(\frac{\partial l'_1}{\partial p_{23}}\right)_o = 0, & \left(\frac{\partial l'_2}{\partial p_{23}}\right)_o = -1, & \left(\frac{\partial q'_1}{\partial p_{23}}\right)_o = 0, & \left(\frac{\partial q'_2}{\partial p_{23}}\right)_o = 0, \\
\left(\frac{\partial l'_1}{\partial p_{33}}\right)_o = l_1, & \left(\frac{\partial l'_2}{\partial p_{33}}\right)_o = l_2, & \left(\frac{\partial q'_1}{\partial p_{33}}\right)_o = 0, & \left(\frac{\partial q'_2}{\partial p_{33}}\right)_o = 0, \\
\left(\frac{\partial l'_1}{\partial \alpha_1}\right)_o = 0, & \left(\frac{\partial l'_2}{\partial \alpha_1}\right)_o = 0, & \left(\frac{\partial q'_1}{\partial \alpha_1}\right)_o = -1, & \left(\frac{\partial q'_2}{\partial \alpha_1}\right)_o = 0, \\
\left(\frac{\partial l'_1}{\partial \alpha_2}\right)_o = 0, & \left(\frac{\partial l'_2}{\partial \alpha_2}\right)_o = 0, & \left(\frac{\partial q'_1}{\partial \alpha_2}\right)_o = 0, & \left(\frac{\partial q'_2}{\partial \alpha_2}\right)_o = -1, \\
\left(\frac{\partial l'_1}{\partial \alpha_3}\right)_o = 0, & \left(\frac{\partial l'_2}{\partial \alpha_3}\right)_o = 0, & \left(\frac{\partial q'_1}{\partial \alpha_3}\right)_o = l_1, & \left(\frac{\partial q'_2}{\partial \alpha_3}\right)_o = l_2.
\end{array}
$$

So, the matrix of the coefficients of the infinitesimal transformations of H_{12} is given by

$$
\zeta_{ij} = \begin{pmatrix}
b_1 & 0 & 0 & c_1 & 0 & 0 & -l_1 & 0 & -q_1 & 0 \\
b_2 & 0 & 0 & c_2 & 0 & 0 & 0 & -l_1 & 0 & -q_1 \\
b_3 & 0 & 0 & c_3 & 0 & 0 & l_1^2 & l_1l_2 & l_1q_1 & l_2q_1 \\
0 & b_1 & 0 & 0 & c_1 & 0 & -l_2 & 0 & -q_2 & 0 \\
0 & b_2 & 0 & 0 & c_2 & 0 & 0 & -l_2 & 0 & -q_2 \\
0 & b_3 & 0 & 0 & c_3 & 0 & l_1l_2 & l_2^2 & l_1q_2 & l_2q_2 \\
0 & 0 & b_1 & 0 & 0 & c_1 & -1 & 0 & 0 & 0 \\
0 & 0 & b_2 & 0 & 0 & c_2 & 0 & -1 & 0 & 0 \\
0 & 0 & b_3 & 0 & 0 & c_3 & l_1 & l_2 & 0 & 0 \\
b_1^2 & b_2b_1 & b_3b_1 & c_1^2 & c_2c_1 & c_3c_1 & 0 & 0 & -1 & 0 \\
b_1b_2 & b_2^2 & b_3b_2 & c_1c_2 & c_2^2 & c_3c_2 & 0 & 0 & 0 & -1 \\
b_1b_3 & b_2b_3 & b_3^2 & c_1c_3 & c_2c_3 & c_3^2 & 0 & 0 & l_1 & l_2
\end{pmatrix}. \qquad (8)
$$

Our aim is to find functions $\Phi(b_1, b_2, b_3, c_1, c_2, c_3, l_1, l_2, q_1, q_2)$ which satisfy the following (Deltheil) system:

$$
\begin{aligned}
b_1\frac{\partial\Phi}{\partial b_1} + c_1\frac{\partial\Phi}{\partial c_1} + (-l_1)\frac{\partial\Phi}{\partial l_1} + (-q_1)\frac{\partial\Phi}{\partial q_1} &= 0\\
b_2\frac{\vartheta\Phi}{\vartheta b_1} + c_2\frac{\vartheta\Phi}{\vartheta c_1} + (-l_1)\frac{\partial\Phi}{\partial l_2} + (-q_1)\frac{\partial\Phi}{\partial q_2} &= 0\\
b_3\frac{\partial\Phi}{\partial b_1} + c_3\frac{\partial\Phi}{\partial c_1} + l_1^2\frac{\partial\Phi}{\partial l_1} + l_1l_2\frac{\partial\Phi}{\partial l_2} + l_1q_1\frac{\partial\Phi}{\partial q_1} + l_2q_1\frac{\partial\Phi}{\partial q_2} &= -4l_1\Phi\\
b_1\frac{\partial\Phi}{\partial b_2} + c_1\frac{\partial\Phi}{\vartheta c_2} + (-l_2)\frac{\partial\Phi}{\partial l_1} + (-q_2)\frac{\partial\Phi}{\partial q_1} &= 0\\
b_2\frac{\partial\Phi}{\partial b_2} + c_2\frac{\partial\Phi}{\partial c_2} + (-l_2)\frac{\partial\Phi}{\partial l_2} + (-q_2)\frac{\partial\Phi}{\partial q_2} &= 0\\
b_3\frac{\partial\Phi}{\partial b_2} + c_3\frac{\partial\Phi}{\partial c_2} + l_1l_2\frac{\partial\Phi}{\partial l_1} + l_2^2\frac{\partial\Phi}{\partial l_2} + l_1q_2\frac{\partial\Phi}{\partial q_1} + l_2q_2\frac{\partial\Phi}{\partial q_2} &= -4l_2\Phi\\
b_1\frac{\partial\Phi}{\partial b_3} + c_1\frac{\partial\Phi}{\partial c_3} + (-\frac{\partial\Phi}{\partial l_1}) &= 0 \qquad (9)\\
b_2\frac{\partial\Phi}{\partial b_3} + c_2\frac{\partial\Phi}{\partial c_3} + (-\frac{\partial\Phi}{\partial l_2}) &= 0\\
b_3\frac{\partial\Phi}{\partial b_3} + c_3\frac{\partial\Phi}{\partial c_3} + l_1\frac{\partial\Phi}{\partial l_1} + l_2\frac{\partial\Phi}{\partial l_2} &= -4\Phi\\
b_1^2\frac{\partial\Phi}{\partial b_1} + b_1b_2\frac{\partial\Phi}{\partial b_2} + b_1b_3\frac{\partial\Phi}{\partial b_3} + c_1^2\frac{\partial\Phi}{\partial c_1} + c_1c_2\frac{\partial\Phi}{\partial c_2} + c_1c_3\frac{\partial\Phi}{\partial c_3} + (-\frac{\partial\Phi}{\partial q_1}) &= -4(b_1 + c_1)\Phi\\
b_1b_2\frac{\partial\Phi}{\partial b_1} + b_2^2\frac{\partial\Phi}{\partial b_2} + b_2b_3\frac{\partial\Phi}{\partial b_3} + c_1c_2\frac{\partial\Phi}{\partial c_1} + c_2^2\frac{\partial\Phi}{\partial c_2} + c_2c_3\frac{\partial\Phi}{\partial c_3} + (-\frac{\partial\Phi}{\partial q_2}) &= -4(b_2 + c_2)\Phi\\
b_1b_3\frac{\partial\Phi}{\partial b_1} + b_2b_3\frac{\partial\Phi}{\partial b_2} + b_3^2\frac{\partial\Phi}{\partial b_3} + c_1c_3\frac{\partial\Phi}{\partial c_1} + c_2c_3\frac{\partial\Phi}{\partial c_2} + c_3^2\frac{\partial\Phi}{\partial c_3} + l_1\frac{\partial\Phi}{\partial q_1} + l_2\frac{\partial\Phi}{\partial q_2} &= -4(b_3 + c_3)\Phi
\end{aligned}
$$

System (9) has $\Phi = 0$ as the trivial solution, obviously. Then by dividing any equation of (12) by Φ, it becomes a (linear non-homogeneous) system of 12 algebraic equations with *ten* unknown quantities:

$$\frac{\partial \ln\Phi}{\partial b_1}, \frac{\partial \ln\Phi}{\partial b_2}, \frac{\partial \ln\Phi}{\partial b_3}, \frac{\partial \ln\Phi}{\partial c_1}, \frac{\partial \ln\Phi}{\partial c_2}, \frac{\partial \ln\Phi}{\partial c_3}, \frac{\partial \ln\Phi}{\partial l_1}, \frac{\partial \ln\Phi}{\partial l_2}, \frac{\partial \ln\Phi}{\partial q_1}, \frac{\partial \ln\Phi}{\partial q_2}.$$

The incomplete and complete matrix (respectively) of the previous system are given by:

$$
\begin{pmatrix}
b_1 & 0 & 0 & c_1 & 0 & 0 & -l_1 & 0 & -q_1 & 0\\
b_2 & 0 & 0 & c_2 & 0 & 0 & 0 & -l_1 & 0 & -q_1\\
b_3 & 0 & 0 & c_3 & 0 & 0 & l_1^2 & l_1l_2 & l_1q_1 & l_2q_1\\
0 & b_1 & 0 & 0 & c_1 & 0 & -l_2 & 0 & -q_2 & 0\\
0 & b_2 & 0 & 0 & c_2 & 0 & 0 & -l_2 & 0 & -q_2\\
0 & b_3 & 0 & 0 & c_3 & 0 & l_1l_2 & l_2^2 & l_1q_2 & l_2q_2\\
0 & 0 & b_1 & 0 & 0 & c_1 & -1 & 0 & 0 & 0\\
0 & 0 & b_2 & 0 & 0 & c_2 & 0 & -1 & 0 & 0\\
0 & 0 & b_3 & 0 & 0 & c_3 & l_1 & l_2 & 0 & 0\\
b_1^2 & b_2b_1 & b_3b_1 & c_1^2 & c_2c_1 & c_3c_1 & 0 & 0 & -1 & 0\\
b_2b_1 & b_2^2 & b_3b_2 & c_2c_1 & c_2^2 & c_3c_2 & 0 & 0 & 0 & -1\\
b_3b_1 & b_3b_2 & b_3^2 & c_3c_1 & c_3c_2 & c_3^2 & 0 & 0 & l_1 & l_2
\end{pmatrix}
$$

$$\begin{pmatrix}
b_1 & 0 & 0 & c_1 & 0 & 0 & -l_1 & 0 & -q_1 & 0 & 0 \\
b_2 & 0 & 0 & c_2 & 0 & 0 & 0 & -l_1 & 0 & -q_1 & 0 \\
b_3 & 0 & 0 & c_3 & 0 & 0 & l_1^2 & l_1 l_2 & l_1 q_1 & l_2 q_1 & -4l_1 \\
0 & b_1 & 0 & 0 & c_1 & 0 & -l_2 & 0 & -q_2 & 0 & 0 \\
0 & b_2 & 0 & 0 & c_2 & 0 & 0 & -l_2 & 0 & -q_2 & 0 \\
0 & b_3 & 0 & 0 & c_3 & 0 & l_1 l_2 & l_2^2 & l_1 q_2 & l_2 q_2 & -4l_2 \\
0 & 0 & b_1 & 0 & 0 & c_1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & b_2 & 0 & 0 & c_2 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & b_3 & 0 & 0 & c_3 & l_1 & l_2 & 0 & 0 & -4 \\
b_1^2 & b_2 b_1 & b_3 b_1 & c_1^2 & c_2 c_1 & c_3 c_1 & 0 & 0 & -1 & 0 & -4(b_1+c_1) \\
b_2 b_1 & b_2^2 & b_3 b_2 & c_2 c_1 & c_2^2 & c_3 c_2 & 0 & 0 & 0 & -1 & -4(b_2+c_2) \\
b_3 b_1 & b_3 b_2 & b_3^2 & c_3 c_1 & c_3 c_2 & c_3^2 & 0 & 0 & l_1 & l_2 & -4(b_3+c_3)
\end{pmatrix}.$$

We consider the 10×10 submatrix of the incomplete matrix which is obtained by deleting the ninth and the twelfth rows. Its determinant is not zero. Therefore, the incomplete matrix has rank 10. As that submatrix is also contained in the complete matrix, adding first the ninth row and then the twelfth row (always considering the last column, obviously), we obtain two 11×11 submatrices.Their determinants are both zero; therefore the complete matrix has rank 10.

We conclude that system (9) is solvable, so there exsists only one not trivial solution given by the function

$$\Phi = k(\sigma_2\rho_1 - \sigma_1\rho_2)^{-4} \quad \text{with } k \in R^*$$

where $\sigma_1 = b_1q_1 + b_2q_2 - 1$, $\rho_2 = c_1q_1 + c_2q_2 - 1$, $\sigma_1 = l_1b_1 + l_2b_2 + b_3$, $\sigma_2 = l_1c_1 + l_2c_2 + c_3$.

We leave out the calculus.

So group H_{12} associated to G_{12} is measurable by Theorem 2. Hence family $\Im_{10}$ is measurable and its density is given by

$$d\Phi = (\sigma_2\rho_1 - \sigma_1\rho_2)^{-4} db_1 \wedge db_2 \wedge db_3 \wedge dc_1 \wedge dc_2 \wedge dc_3 \wedge dl_1 \wedge dl_2 \wedge dq.$$

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Additive Closure Operators on Abelian Unital *l*-groups

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Abstract

In the paper an additive closure operator on an abelian unital l-group (G, u) is introduced and one studies the mutual relation of such operators and of additive closure ones on the MV-algebra $\Gamma(G, u)$.

Key words: MV-algebra; l-group.

2000 Mathematics Subject Classification: 06D35, 06F20

1 Introduction

In [6] additive closure (and multiplicative interior) operators on MV-algebras were introduced as a natural generalization of topological closure (and interior) operators on Boolean algebras. Closure and interior MV-algebras (MV-algebras endowed with additive closure or multiplicative interior operators) generalize topological boolean algebras in a natural way.

Let us recall the notions of an MV-algebra and of an additive closure operator on an MV-algebra.

Definition 1.1 An algebra $\mathcal{A} = (A, \oplus, \neg, 0)$ of the signature $\langle 2, 1, 0 \rangle$ is called *an MV-algebra* iff for each $x, y, z \in A$:

(MV1) $\quad x \oplus (y \oplus z) = (x \oplus y) \oplus z$;

(MV2) $\quad x \oplus y = y \oplus x$;

(MV3) $x \oplus 0 = x$;

(MV4) $\neg\neg x = x$;

(MV5) $x \oplus \neg 0 = \neg 0$;

(MV6) $\neg(\neg x \oplus y) \oplus y = \neg(x \oplus \neg y) \oplus x$.

Definition 1.2 Let us consider an MV-algebra $\mathcal{A} = (A, \oplus, \neg, 0)$ and a mapping $Cl : A \to A$. Then Cl is called *an additive closure operator* on $\mathcal{A}$ iff for each $a, b \in A$

1. $Cl(a \oplus b) = Cl(a) \oplus Cl(b)$,
2. $a \leq Cl(a)$,
3. $Cl(Cl(a)) = Cl(a)$,
4. $Cl(0) = 0$.

MV-algebras, which are an algebraic counterpart of the Łukasiewicz infinite valued logic, are by [3], Chapters 2, 7 in a very close connection with abelian unital l-groups.

Definition 1.3 An algebra $G = (G, +, 0, \vee, \wedge)$ of the signature $\langle 2, 0, 2, 2\rangle$ is called *an l-group* iff

1. $(G, +, 0)$ is a group,
2. $(G, \vee, \wedge)$ is a lattice,
3. $x + (y \vee z) + w = (x + y + w) \vee (x + z + w) \qquad \forall x, y, z, w \in G,$
   $x + (y \wedge z) + w = (x + y + w) \wedge (x + z + w) \qquad \forall x, y, z, w \in G.$

An element $u \in G$ $(u > 0)$ is called *a strong unit* of the l-group G iff

$$(\forall a \in G)(\exists n \in \mathbb{N})\ (a \leq nu),$$

where

$$nu \overset{\text{def}}{=} \underbrace{u + u + \cdots + u}_{n}.$$

If an l-group G contains a strong unit u, then (G, u) is called *a unital l-group*. Moreover, if the operation "+“ of the l-group G is commutative, then G is called *an abelian l-group*.

In the following remark we will describe the mutual relation of abelian unital l-groups and MV-algebras.

Remark 1.4

a) Let $(G, +, 0, \vee, \wedge)$ be an abelian l-group and let $u \in G, u \geq 0$. If

$$x \oplus y := (x + y) \wedge u, \qquad \neg x := u - x,$$

then $\Gamma(G, u) = ([0, u], \oplus, \neg, 0, u)$ is an MV-algebra.

b) On the other hand, Daniele Mundici [5] proved that for every MV-algebra $\mathcal{A}$ there exists such an abelian unital l-group (G, u) that $\mathcal{A} \cong \Gamma(G, u)$.

The aim of this paper is to introduce an additive closure operator on an abelian unital l-group (G, u). That means, we will investigate in introducing of such an operator on abelian unital l-groups that it will preferably form a natural counterpart of additive closure operators on MV-algebras.

2 Relation between additive closure operators on MV-algebras and on abelian unital l-groups

Definition 2.1 Let (G, u) be an abelian unital l-group. A mapping $\psi^+ : G^+ \to G^+$ such that for each $x, y \in G^+$ it holds

1. $\psi^+(x + y) = \psi^+(x) + \psi^+(y)$,
2. $\psi^+(x \wedge u) = \psi^+(x) \wedge u$,
3. $x \leq \psi^+(x)$,
4. $\psi^+(\psi^+(x)) = \psi^+(x)$,

will be called *an additive closure operator on* G^+, where $G^+ = \{x \in G;\ x \geq 0\}$.

Lemma 2.2 *Let (G, u) be an abelian unital l-group and let ψ^+ be an additive closure operator on G^+. Then we have for each $k \in \mathbb{N}$, $k > 1$ and for each $x, y \in G^+$*

(i) $\psi^+(u) = u$,

(ii) $\psi^+(ku) = ku$,

(iii) $x \leq y \Rightarrow \psi^+(x) \leq \psi^+(y)$.

Proof

(i) From the axiom 3 of Definition 2.1 it follows that $u \leq \psi^+(u)$. Moreover, from the second axiom of the same definition we get

$$\psi^+(u) = \psi^+(u \wedge u) = \psi^+(u) \wedge u$$

and further $\psi^+(u) \leq u$. Together we have $u = \psi^+(u)$.

(ii) It follows from the first axiom of Definition 2.1 and from (i).

(iii) Let $x, y \in G^+, x \leq y$. Since $-x + (x \vee y) \in G^+$, it must also be

$$\psi^+(y) = \psi^+(x \vee y) = \psi^+(x + (-x + (x \vee y))) = \psi^+(x) + \psi^+(-x + (x \vee y)),$$

But since

$$\psi^+(-x + (x \vee y)) \in G^+,$$

we finally get

$$\psi^+(x) \leq \psi^+(y).$$

□

Definition 2.3 Let (G, u) be an abelian unital l-group. A mapping $\psi : G \to G$ is called *an additive closure operator on* G iff there exists such an additive closure operator ψ^+ on G^+, that it holds for each element $a \in G$

1. $\psi \mid_{G^+} = \psi^+$,
2. $\psi(a) = \psi^+(a^+) - \psi^+(a^-)$, where $a^+ = a \vee 0$, $a^- = -a \vee 0$.

Remark 2.4 It is known that in each l-group G we have $a = a^+ - a^-$ for each element $a \in G$. So $G = G^+ - G^+$ holds in each l-group G. Let us show now that in each l-group G all representations of $\psi(a)$ in the form of the difference of $\psi^+(x)$ and $\psi^+(y)$, where $x, y \in G^+$ such that $a = x - y$, are the same as the representation of $\psi(a)$ in the form of the difference of $\psi^+(a^+)$ and $\psi^+(a^-)$.

Lemma 2.5 *Let* (G, u) *be an abelian unital* l*-group and let* ψ *be an additive closure operator on* G*. Then it holds for each element* $a \in G$ *and for each elements* $x, y \in G^+$

$$[a = x - y] \Longrightarrow [\psi(a) = \psi^+(a^+) - \psi^+(a^-) = \psi^+(x) - \psi^+(y)].$$

Proof If $a = x - y$, then $x - y = a^+ - a^-$. From that we have $x + a^- = a^+ + y$ and so $\psi^+(x) + \psi^+(a^-) = \psi^+(a^+) + \psi^+(y)$, and finally $\psi^+(x) - \psi^+(y) = \psi^+(a^+) - \psi^+(a^-) = \psi(a)$. □

In the sequel we will study the mutual relation of additive closure operators on abelian unital l-groups and on MV-algebras. The properties of additive closure operators on MV-algebras were studied in [6].

Theorem 2.6 *Let us consider an abelian unital* l*-group* (G, u) *and further an additive closure operator* ψ^+ *on* G^+*. Then* $\varphi = \psi^+ \mid_{[0,u]}$ *is an additive closure operator on the* MV*-algebra* $\mathcal{A} = \Gamma(G, u)$*.*

Proof Since ψ^+ is isotone and $\psi^+(u) = u$, it is obvious that φ is a mapping from $[0, u]$ into $[0, u]$. We will check now validity of 1.–4. from Definition 1.2. Therefore, let us choose two arbitrary elements $a, b \in [0, u]$ and we have

1. $\varphi(a \oplus b) = \varphi((a + b) \wedge u) = \psi^+((a + b) \wedge u) = \psi^+(a + b) \wedge u = (\psi^+(a) + \psi^+(b)) \wedge u = (\varphi(a) + \varphi(b)) \wedge u = \varphi(a) \oplus \varphi(b)$,
2. $a \le \psi^+(a) = \varphi(a)$,
3. $\varphi(\varphi(a)) = \psi^+(\varphi(a)) = \psi^+(\psi^+(a)) = \psi^+(a) = \varphi(a)$,
4. $\varphi(0) = \psi^+(0) = 0$, because of $\psi^+(0) = \psi^+(0 + 0) = \psi^+(0) + \psi^+(0)$. □

Let $\mathcal{A} = \Gamma(G, u)$ be the MV-algebra constructed on an abelian unital l-group (G, u). Then by [3], Lemma 7.1.3 each element $a \in G^+$ can be uniquely represented in the form

$$a = a_1 + a_2 + \cdots + a_n,$$

where the n-tuple $(a_1, a_2, \ldots, a_n) \in [0, u]^n$ is determined by relations

$$a_1 = a \wedge u,\ a_2 = (a - a_1) \wedge u, \ldots,\ a_n = (a - a_1 - \cdots - a_{n-1}) \wedge u.$$

Remark 2.7 The introduced n-tuple $(a_1, a_2, \dots, a_n)$ is *a good sequence* of elements of MV-algebra $\Gamma(G, u)$—see [3, Lemma 7.1.3]. Let us recall that *a good sequence* of elements of an MV-algebra $\mathcal{A}$ is such a sequence $(a_1, a_2, \dots, a_n, \dots)$ of elements of this algebra that for each $i = 1, 2, \dots$ the identity

$$a_i \oplus a_{i+1} = a_i$$

holds and at the same time there exists such $n \in \mathbb{N}$ that $a_r = 0$ for all $r > n$.

Now, let φ be an additive closure operator on the MV-algebra $\mathcal{A} = \Gamma(G, u)$ and let us define a mapping $\overline{\varphi} : G^+ \to G^+$, where

$$\overline{\varphi}(a) \stackrel{\text{def}}{=} \varphi(a_1) + \varphi(a_2) + \cdots + \varphi(a_n) \qquad \forall a \in G^+.$$

Remark 2.8 Let us notice the prescription for the introduced mapping $\overline{\varphi}$. By Remark 2.5 we know that $(a_1, a_2, \dots, a_n)$ is a good sequence of elements of $\Gamma(G, u)$ and for each $i = 1, 2, \dots, n-1$ we have therefore $a_i \oplus a_{i+1} = a_i$. But then also for each $i = 1, 2, \dots, n-1$

$$\varphi(a_i) \oplus \varphi(a_{i+1}) = \varphi(a_i \oplus a_{i+1}) = \varphi(a_i).$$

That means, $(\varphi(a_1), \varphi(a_2), \dots, \varphi(a_n))$ is a good sequence of elements of $\Gamma(G, u)$ again.

Lemma 2.9 *Let us consider an MV-algebra $\mathcal{A} = \Gamma(G, u)$ constructed on an abelian unital l-group (G, u) and an additive closure operator φ on $\mathcal{A}$. Then the mapping $\overline{\varphi}$ is isotone.*

Proof Let us choose arbitrary elements $a, b \in G^+$, $a \le b$. It holds ([3, Lemma 7.1.3])

$$a = a_1 + a_2 + \cdots + a_m, \qquad b = b_1 + b_2 + \cdots + b_n,$$

where $a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_n \in [0, u]$ and m, n are some integers, not necessarily the same. If for example $m > n$, then we put $b_{m-n+1} = \cdots = b_m = 0$. So we can consider $m = n$. Now, if $a \le b$, then for each integer k

$$((a - ku) \vee 0) \wedge u \le ((b - ku) \vee 0) \wedge u.$$

Further by [3, Lemma 7.1.3] we have from the last inequality

$$(a - a_1 - a_2 - \cdots - a_k) \wedge u \le (b - b_1 - b_2 - \cdots - b_k) \wedge u,$$

that means $a_{k+1} \le b_{k+1}$ for each integer k. From that it follows that $\varphi(a_{k+1}) \le \varphi(b_{k+1})$ for each integer k and finally

$$\overline{\varphi}(a) = \varphi(a_1) + \varphi(a_2) + \cdots + \varphi(a_n) \le \varphi(b_1) + \varphi(b_2) + \cdots + \varphi(b_n) = \overline{\varphi}(b).$$

Theorem 2.10 *Let $\mathcal{A} = \Gamma(G, u)$ be the MV-algebra constructed on an abelian unital l-group (G, u) and let φ be an additive closure operator on $\mathcal{A}$. Then for the mapping $\overline{\varphi}$ and an arbitrary element $a \in G^+$*

- $\overline{\varphi}(a \wedge u) = \overline{\varphi}(a) \wedge u$,
- $a \leq \overline{\varphi}(a)$,
- $\overline{\varphi}(\overline{\varphi}(a)) = \overline{\varphi}(a)$.

Proof Let $a \in G^+$ is chosen arbitrarily. Then there exists an n-tuple $(a_1, a_2, \dots, a_n)$ of elements from $[0, u]$, where $a = a_1 + a_2 + \cdots + a_n$, $a_1 = a \wedge u$, $a_2 = (a - a_1) \wedge u$, ..., $a_n = (a - a_1 - \cdots - a_{n-1}) \wedge u$. We have:

- $\overline{\varphi}(a) \wedge u = (\varphi(a_1) + \varphi(a_2) + \cdots + \varphi(a_n)) \wedge u = \varphi(a_1) \oplus \varphi(a_2) \oplus \cdots \oplus \varphi(a_n) = = \varphi(a_1 \oplus a_2 \oplus \cdots \oplus a_n) = \varphi((a_1 + a_2 + \cdots + a_n) \wedge u) = \varphi(a \wedge u) = \overline{\varphi}(a \wedge u)$;
- $a = a_1 + a_2 + \cdots + a_n \leq \varphi(a_1) + \varphi(a_2) + \cdots + \varphi(a_n) = \overline{\varphi}(a)$;
- $\overline{\varphi}(\overline{\varphi}(a)) = \overline{\varphi}(\varphi(a_1) + \varphi(a_2) + \cdots + \varphi(a_n)) = \varphi(\varphi(a_1)) + \varphi(\varphi(a_2)) + \cdots + \varphi(\varphi(a_n)) = \varphi(a_1) + \varphi(a_2) + \cdots + \varphi(a_n) = \overline{\varphi}(a)$, because of $c = \varphi(a_1) + \varphi(a_2) + \cdots + \varphi(a_n)$ is just the unique decomposition of the element $c \in G^+$ onto a sum of elements from $[0, u]$, which form a good sequence of $\Gamma(G, u)$.

□

Remark 2.11 (open problem) In Theorem 2.10, we have proven in fact that the operator $\overline{\varphi}$ fulfils conditions 2, 3 and 4 from Definition 2.1. Not answered stays now the problem, in which condition does $\overline{\varphi}$ fulfil moreover the axiom 1 from Definition 2.1, that means in which condition does $\overline{\varphi}$ become an additive closure operator on G^+.

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Uncertainty of Coordinates and Looking for Dispersion of GPS Receiver

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Abstract

The aim of the paper is to show some possible statistical solution of the estimation of the dispersion of the GPS receiver. The presented method (based on theory of linear model with additional constraints of type I) can serve for an improvement of the accuracy of estimators of coordinates acquired from the GPS receiver.

Key words: Two stage regression models; BLUE; uncertainty of the type A and B; confidence ellipsoids; variance components.

2000 Mathematics Subject Classification: 62J05, 62J10, 62F10

1 Introduction

The aim of this paper is to make one keep in view that the geographical coordinates, obtained with the help of a GPS receiver cannot be regarded as accurate data. Based on the results of one exemplary measurement, we will show that it is always necessary to take into account an uncertainty of data acquired from the GPS receiver. The user of the GPS receiver should always consider carefully if the measured values are sufficiently accurate with respect to the particular purposes. This conclusion can be drawn only in cases when an estimation of a dispersion of the GSP receiver is known in a given place and time.

In order to lower the uncertainty of the measurement, various measuring approaches are used. A repeated (multistage) measurement is one of such procedures. In addition, it is also well-known how to determine the estimation of the dispersion of the GPS receiver.

However, a possible situation can arise when the user of the device is not in a position to repeat the measurement several times during longer time interval. This can be caused either by a physical principle of a given design of the measurement or by practical aspects (e.g. expensiveness of the repeated measurement carried out for several days).

To avoid this difficulty, we will show another possible approach which leads to the estimation of the dispersion of the GPS receiver. Moreover, the presented method can serve for an improvement of the accuracy of data acquired from the GPS receiver.

In the following text, an algorithm based on the theory of estimation is introduced which would eventually decrease the uncertainty of the coordinates obtained from the GPS receiver with an utilization of an additional measurement (in our case, by a measuring tape). Even for an amateur measurement, the dispersion of the measured lengths is approximately about 0.1^2 m^2. From here and on, the uncertainty of the first-stage measurement is considered as the B-type uncertainty (in our case, the B-type uncertainty represents the uncertainty of the measurement by the measuring tape) and the lengths obtained in the first-stage measurement are denoted by a symbol Θ. On the contrary, the uncertainty of the second-stage mesurement is considered as the A-type uncertainty (in our case, the A-type uncertainty represents the uncertainty of the measurement by the GPS receiver) and the coordinates acquired in the second-stage measurement are denoted by a symbol β.

Motivation

Let us suppose the following situation. The goal was to determine a stochastic distribution of a chemical element in the soil. The coordinates of the positions, where the value of the chemical element was intended to be measured, have been acquired by the GPS receiver. The obtained values are depicted in Figure 1 where every point corresponds to the place where the sample was taken. According to the design of the measurement and principle of the utilized device, it was then expected that the acquired data would create an accumulation in the form of a ring.

As it is evident from Figure 1, the ring was generated from data for one “locality”. However, the expected ring for the second locality was extended in comparison with the previous one. One may therefore ask the following questions. What were the reasons for such an anomalous behaviour of the measured data? Was it a consequence of the uncertainty of the acquired coordinates?

In the next example from another area of interest, it will be shown that the estimation of the dispersion of the GPS device is 0.354^2 m^2. This value may greatly differ depending on a number of available satellites, surrounding landscape and sedulity of the person performing the measurement. Therefore, the values acquired by the GPS receiver can exhibit different accuracy.

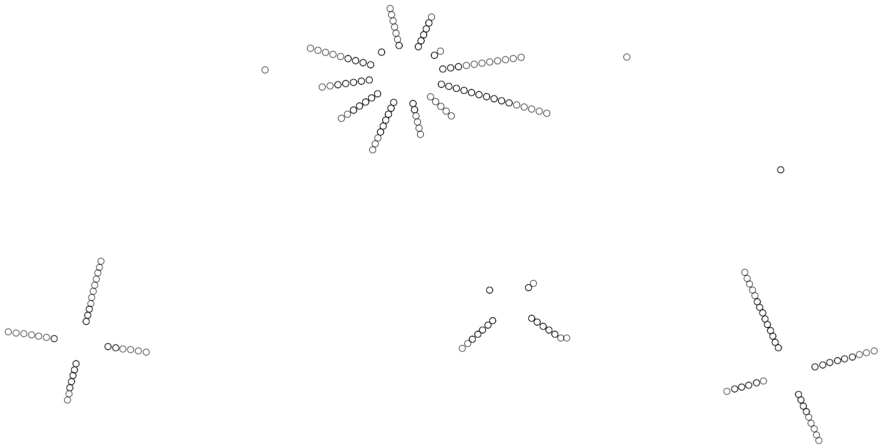


Figure 1: Coordinates of the measured points.

In the above-discussed example describing the measurement of the location points in the soil, it was found out that the student carrying out the measurement did not respect the instructions for a given measurement. The measurement was not performed all at once but there was a time delay between particular steps of the measurement.

Notation

The following notations will be used throughout the paper:

$\mathbb{R}^n$	space of all n-dimensional real vectors;
Θ	real column vector—from the first stage;
β	real column vector—from the second stage;
$\mathbf{I}_{m,m}$, $\mathbf{A}_{m,n}$	$m \times m$ identity matrix; real $m \times n$ matrix;
$\mathbf{A}_{r_1:s_1,r_2:s_2}$	$(s_1 - r_1) \times (s_2 - r_2)$ block matrix with elements of $\mathbf{A}$;
$\mathbf{A}', r(\mathbf{A}), \mathrm{Tr}(\mathbf{A})$	transpose, rank and trace of the matrix $\mathbf{A}$;
$\mathbf{A} = \mathrm{diag}(\mathbf{u})$	diagonal matrix with diagonal equal elements of vector $\mathbf{u}$;
$\mathcal{M}(\mathbf{A})$	column space of the matrix $\mathbf{A}$; $\mathcal{M}(\mathbf{A}) = \{\mathbf{Au} : \mathbf{u} \in \mathbb{R}^n\} \subset \mathbb{R}^m$;
$\mathrm{Ker}(\mathbf{A})$	null space of the matrix $\mathbf{A}$; $\mathrm{Ker}(\mathbf{A}) = \{\mathbf{u} : \mathbf{u} \in \mathbb{R}^n, \mathbf{Au} = \mathbf{0}\} \subset \mathbb{R}^n$;
$\mathbf{A}^-$	generalized inverse of the matrix $\mathbf{A}$ (satisfying $\mathbf{AA}^-\mathbf{A} = \mathbf{A}$), (see [4]);
$\mathrm{P}_{\mathbf{A}}$	orthogonal projector onto $\mathcal{M}(\mathbf{A})$ in Euclidean norm; $\mathrm{P}_{\mathbf{A}} = \mathbf{A}(\mathbf{A}'\mathbf{A})^-\mathbf{A}'$;
$\mathrm{M}_{\mathbf{A}}$	orthogonal projector onto $\mathcal{M}^{\perp}(\mathbf{A}) = \mathrm{Ker}(\mathbf{A}')$ in Euclidean norm; $\mathrm{M}_{\mathbf{A}} = \mathbf{I} - \mathrm{P}_{\mathbf{A}}$;
$\mathbf{Y} \sim (\mathbf{A}\Theta, \mathbf{T})$	observation vector $\mathbf{Y}$ with mean value $\mathbf{A}\Theta$ and covariance matrix $\mathbf{T}$.

2 Model of measurements

Definition 1 Let us consider the following linear model $\mathbf{Y}-\mathbf{D}\hat{\Theta} \sim_n (\mathbf{X}\beta, \Sigma_0)$, where $\mathbf{\Sigma}_0 = \sigma^2\mathbf{V}_1 + \mathbf{D}\mathbf{V}_0\mathbf{D}'$ and where $\mathbf{Y} \sim_n (\mathbf{D}\Theta + \mathbf{X}\beta, \sigma^2\mathbf{V}_1)$ is a random observation vector, $\beta \in \mathbb{R}^k$ stands for a vector of the useful parameters and $\mathbf{X}_{n,k}$ denotes a design matrix belonging to the vector β. We suppose that an estimator $\widehat{\Theta} \sim_{k_1} (\Theta, \mathbf{V}_0)$ of Θ is at our disposal only.

Theorem 1 *The standard estimator $\hat{\sigma}^2$ of the parameter σ^2 for the model defined in Definition 1 is given by the expression in the form of*

$$\begin{aligned}\hat{\sigma}^2 = \ &\lambda[(\mathbf{Y}-\mathbf{D}\widehat{\Theta})'(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}})^+\mathbf{V}_1(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}})^+(\mathbf{Y}-\mathbf{D}\widehat{\Theta})] \\ &-\mathrm{Tr}[(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}})^+\mathbf{V}_1(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}})^+\mathbf{V}_0]\,,\end{aligned}$$

where the value of the parameter λ is expressed by the following equation

$$\mathbf{S}_{(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}})^+}\lambda = 1\,,$$

where the 1×1 matrix $\mathbf{S}_{(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}})^+}$ takes the form of

$$\mathbf{S}_{(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}})^+} = \mathrm{Tr}[(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}})^+\mathbf{V}_1(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}})^+\mathbf{V}_1]\,.$$

Proof Firstly, we have the model in the form of

$$\mathbf{Y} \sim (\mathbf{X}\beta, \mathbf{V}_0 + \sigma^2\mathbf{V}_1),$$

where the estimator $\hat{\sigma}^2$ of the parameter σ^2 takes the form of

$$\hat{\sigma}^2 = \mathbf{Y}'\mathbf{A}\mathbf{Y} + \mathbf{a},$$

where $\mathbf{A}$ is a suitable matrix. Let $\mathbf{E}[\hat{\sigma}^2] = \sigma^2$, which is equivalent to $\mathbf{E}[\hat{\sigma}^2] = \mathrm{Tr}(\mathbf{A}\mathbf{V}_0) + \sigma^2\,\mathrm{Tr}(\mathbf{A}\mathbf{V}_1) + \beta'\mathbf{X}'\mathbf{A}\mathbf{X}\beta + \mathbf{a} = \sigma^2$. This implies that $\mathbf{a} = -\,\mathrm{Tr}(\mathbf{A}\mathbf{V}_0)$, $\mathrm{Tr}(\mathbf{A}\mathbf{V}_1) = 1$ and $\mathbf{X}'\mathbf{A}\mathbf{X} = 0$.

It is known (see [1]) that the matrix $\mathbf{A}$ in the form of $\mathbf{A} = \mathrm{M}_{\mathbf{X}}\mathbf{S}\mathrm{M}_{\mathbf{X}}$, where $\mathbf{S} = \mathbf{S}'$, satisfies the conditions $\mathbf{A}\mathbf{X} = \mathbf{0}$ and $\mathbf{A} = \mathbf{A}'$. This leads to the minimalization of the functional Φ defined as $\Phi = \mathrm{Tr}(\mathbf{A}\mathbf{\Sigma}_0\mathbf{A}\mathbf{\Sigma}_0) - 2\lambda\,\mathrm{Tr}(\mathbf{A}\mathbf{V}_1)$. This can be rewritten as

$$\Phi(\mathbf{S}) = \mathrm{Tr}(\mathbf{S}\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}}\mathbf{S}\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}}) - 2\lambda\,\mathrm{Tr}(\mathbf{S}\mathrm{M}_{\mathbf{X}}\mathbf{V}_1\mathrm{M}_{\mathbf{X}}).$$

As $\frac{\partial\Phi(\mathbf{S})}{\partial\mathbf{S}} = 0$, we arrive at $4(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}})\mathbf{S}(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}}) = 4\lambda\mathrm{M}_{\mathbf{X}}\mathbf{V}_1\mathrm{M}_{\mathbf{X}}$. Now we have the matrix system in the form of $\mathbf{A}\mathbf{X}\mathbf{B} = \mathbf{C}$. The general solution of this matrix system is $\mathbf{X} = \mathbf{A}^-\mathbf{C}\mathbf{B}^- + \mathbf{Z} - \mathbf{A}^-\mathbf{A}\mathbf{Z}\mathbf{B}\mathbf{B}^-$ It is possible to show, that $\mathbf{X} = \mathbf{A}^+\mathbf{C}\mathbf{B}^+$ is also the solution (see [4]).
As $\mathrm{M}_{\mathbf{X}}\mathbf{S}\mathrm{M}_{\mathbf{X}} = \mathbf{A}$, it follows that

$$\mathbf{A} = \lambda(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}})^+\mathbf{V}_1(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}})^+\,.$$

With regard to the condition that $\mathrm{Tr}(\mathbf{A}\mathbf{V}_1) = 1$, we arrive at

$$\lambda\,\mathrm{Tr}[(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}})^+\mathbf{V}_1(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}})^+\mathbf{V}_1] = 1.$$

From this relation, we can obtain the result for the matrix $\mathbf{A}$ and the equation for the Lagrange parameter λ. We then get

$$\lambda = \frac{1}{\mathrm{Tr}[(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}})^{+}\mathbf{V}_1(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}})^{+}\mathbf{V}_1]},$$

$$\mathbf{A} = \frac{(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}})^{+}\mathbf{V}_1(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}})^{+}}{\mathrm{Tr}[(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}})^{+}\mathbf{V}_1(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}})^{+}\mathbf{V}_1]}\,.$$

Finally, the estimator $\hat{\sigma}^2$ of the parameter σ^2 for the matrix $\mathbf{\Sigma}_0$ can be now written as

$$\hat{\sigma}^2 = \mathbf{Y}'\mathbf{A}\mathbf{Y} - \mathrm{Tr}(\mathbf{A}\mathbf{V}_0) = \frac{\mathbf{Y}'(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}})^{+}\mathbf{V}_1(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}})^{+}\mathbf{Y}}{\mathrm{Tr}[(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}})^{+}\mathbf{V}_1(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}})^{+}\mathbf{V}_1]} - \frac{\mathrm{Tr}[(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}})^{+}\mathbf{V}_1(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}})^{+}\mathbf{V}_0]}{\mathrm{Tr}[(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}})^{+}\mathbf{V}_1(\mathrm{M}_{\mathbf{X}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{X}})^{+}\mathbf{V}_1]}. \qquad \square$$

Hereafter we will focus on the same model but from a different point of view. We will consider the model of the measurement and then we will present how to determine the estimators of the fundamental parameters.

Definition 2 The model of connecting measurement will be represented by

$$(i) \qquad \begin{pmatrix}\mathbf{Y}_1\\ \mathbf{Y}_2\end{pmatrix} \sim \left[\begin{pmatrix}\mathbf{X}_1, & \mathbf{0}\\ \mathbf{D}, & \mathbf{X}_2\end{pmatrix}\begin{pmatrix}\Theta\\ \beta\end{pmatrix}, \begin{pmatrix}\mathbf{\Sigma}_{11}, & \mathbf{0}\\ \mathbf{0}, & \mathbf{\Sigma}_{22}\end{pmatrix}\right],$$

where $\mathbf{X}_1, \mathbf{D}, \mathbf{X}_2$ are known $n_1 \times k_1, n_2 \times k_1, n_2 \times k_2$ matrices, respectively, such that $\mathrm{M}(\mathbf{D}') \subset \mathrm{M}(\mathbf{X}_1')$; Θ and β are unknown k_1- and k_2-dimensional vectors; $\mathbf{\Sigma}_{22} = \sigma^2\mathbf{V}_1$, where $\mathbf{\Sigma}_{11}$ and $\mathbf{V}_1$ are known matrices.

In this model, the parameter Θ is estimated on the basis of the vector $\mathbf{Y}_1$ of the first stage and parameter β on the basis of the vectors $\mathbf{Y}_2 - \mathbf{D}\hat{\Theta}$ and $\hat{\Theta}$.

At this point, it should be mentioned that the results of the measurement from the second stage (i.e. $\mathbf{Y}_2$) cannot be used for a modification of the estimator $\hat{\Theta}$.

The parametric space $\underline{\Theta}$ of this model of connecting measurement $\mathbf{Y}$ is defined as

$$(ii) \qquad \underline{\Theta} = \{(\Theta', \beta')' : \mathbf{B}\beta + \mathbf{C}\Theta + \mathbf{a} = \mathbf{0}\}\,,$$

where $\mathbf{B}$ and $\mathbf{C}$ are $q \times k_2$ and $q \times k_1$ matrices, $\mathbf{a}$ is q-dimensional vector, $r(\mathbf{B}) = q < k_2$.

Definition 3 The model in the parametric space $\underline{\Theta}$ (see Definition 2) is regular provided that $r(\mathbf{X}_1) = k_1$, $r(\mathbf{X}_2) = k_2$, $\mathbf{\Sigma}_{11}, \mathbf{\Sigma}_{22}$ are positively definite matrices, $r(\mathbf{B}) = q$.

Remark 1 The vector Θ represents the parameter of the first stage (connecting) whereas the vector β denotes the parameter of the second stage (connected). In the second stage, we then start with the unbiased estimator $\widehat{\Theta} = (\mathbf{X}_1'\mathbf{\Sigma}_{11}^{-1}\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{\Sigma}_{11}^{-1}\mathbf{Y}_1$ originating from the first stage whose covariance matrix is expressed in the form of $\mathrm{Var}(\widehat{\Theta}) = \mathbf{V}_0 = (\mathbf{X}_1'\mathbf{\Sigma}_{11}^{-1}\mathbf{X}_1)^{-1}$.

Definition 4 The least-square estimator of the parameter β, obtained under the condition that $\mathbf{\Sigma}_{11} = \mathbf{0}$ ($\Rightarrow \mathrm{Var}(\widehat{\Theta}) = \mathbf{0}$), is called the standard estimator if the vector Θ is substituted by $\widehat{\Theta}$ in this estimator.

Theorem 2 *The standard estimator $\widehat{\beta}$ of the parameter β in the model (i) and (ii) postulated in Definition 2 and given by*

$$\begin{aligned}\widehat{\beta} = {} & (\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}(\mathbf{Y}_2 - \mathbf{D}\widehat{\Theta}) \\ & - (\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}'[\mathbf{B}(\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}']^{-1} \\ & \times \{\mathbf{a} + \mathbf{C}\widehat{\Theta} + \mathbf{B}(\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}(\mathbf{Y}_2 - \mathbf{D}\widehat{\Theta})\},\end{aligned}$$

is unbiased.

Proof See [3], p. 72–73. □

Theorem 3 *If* $\mathrm{Var}(\widehat{\Theta}) \neq \mathbf{0}$ *then the covariance matrix of the standard estimator $\widehat{\beta}$ is composed of two uncertainties, i.e. the "uncertainty of type A" and "uncertainty of type B", as*

$$\begin{aligned}\mathrm{Var}(\widehat{\beta}) = {} & \underbrace{\mathrm{Var}_0(\widehat{\beta})}_{\substack{\text{uncertainty of}\\ \text{type } A}} \underbrace{\begin{aligned} & + \langle\{\mathbf{I} - (\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}'[\mathbf{B}(\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}']^{-1}\mathbf{B}\} \\ & \times (\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{D} - (\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2)^{-1} \\ & \times \mathbf{B}'[\mathbf{B}(\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}']^{-1}\mathbf{C}\rangle \\ & \times \mathrm{Var}(\widehat{\Theta}) \\ & \times \langle\{\mathbf{I} - (\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}'[\mathbf{B}(\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}']^{-1}\mathbf{B}\} \\ & \times (\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{D} - (\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2)^{-1} \\ & \times \mathbf{B}'[\mathbf{B}(\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}']^{-1}\mathbf{C}\rangle' \end{aligned}}_{\substack{\text{uncertainty of}\\ \text{type } B}}\end{aligned}$$

where

$$\begin{aligned}\mathrm{Var}_0(\widehat{\beta}) = {} & (\mathrm{M}_{\mathbf{B}'}\mathbf{X}_2'\mathbf{\Sigma}_{22}\mathbf{X}_2\mathrm{M}_{\mathbf{B}'})^+ = (\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2)^{-1} - (\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}' \\ & \times[\mathbf{B}(\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}']^{-1}\mathbf{B}(\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2)^{-1}.\end{aligned}$$

Proof See [3], p. 74. □

Corollary 1 *For the case of the model with* $\mathbf{X}_2 = \mathbf{I}$ *and* $\mathbf{D} = \mathbf{0}$*, the covariance matrix of the standard estimator is given by*

$$\begin{aligned}\mathrm{Var}(\widehat{\beta}) = {} & [\mathbf{I} - \mathbf{\Sigma}_{22}\mathbf{B}'(\mathbf{B}\mathbf{\Sigma}_{22}\mathbf{B}')^{-1}\mathbf{B}]\mathbf{\Sigma}_{22}[\mathbf{I} - \mathbf{B}'(\mathbf{B}\mathbf{\Sigma}_{22}\mathbf{B}')^{-1}\mathbf{B}\mathbf{\Sigma}_{22}] \\ & + \mathbf{\Sigma}_{22}\mathbf{B}'(\mathbf{B}\mathbf{\Sigma}_{22}\mathbf{B}')^{-1}\mathbf{C}\,\mathrm{Var}(\hat{\Theta})\mathbf{C}'(\mathbf{B}\mathbf{\Sigma}_{22}\mathbf{B}')^{-1}\mathbf{B}\mathbf{\Sigma}_{22}\,.\end{aligned}$$

Proof See [3], p. 73–74. □

Theorem 4 *The* $(1-\alpha)$*-confidence domain for the parameter* β, $\beta \in \underline{\Theta}$ *(see Definition 2), based on the standard* BLUE $\widehat{\beta}$, *is a set expressed by*

$$\mathcal{E}_{1-\alpha}(\beta) = \left\{ \mathbf{u} : \mathbf{u} \in \underline{\Theta}_\beta \subset \mathbb{R}^{k_2}, (\mathbf{u} - \widehat{\beta})'[\mathrm{Var}(\widehat{\beta})]^-(\mathbf{u} - \widehat{\beta}) \leq \chi^2_{r[\mathrm{Var}(\widehat{\beta})]}(1-\alpha) \right\}.$$

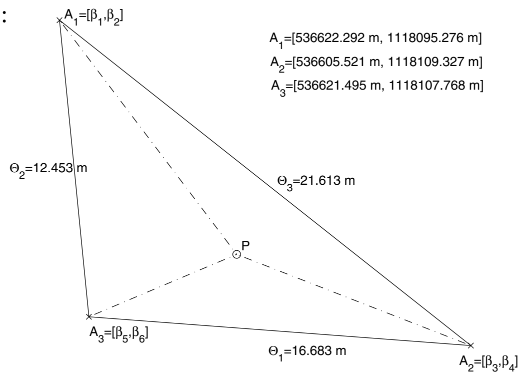
Here the symbol $\chi^2_{r[\mathrm{Var}(\widehat{\beta})]}(1-\alpha)$ *denotes* $(1-\alpha)$*-quantile of* χ^2*-distribution with* $r[\mathrm{Var}(\widehat{\beta})]$ *degrees of freedom.*

Proof See [2], p. 158–159. □

3 Illustrative example

The aim of this example is to find a dispersion for a CARMIN GPS 12XL navigator and estimate the plane coordinates β of the points A_1, A_2, A_3 in the Situation I and plane coordinates of the points A_1, A_2, A_3 and P in the Situation II using the theory of basic linear models of the measurement.

Situation I:



Situation II:

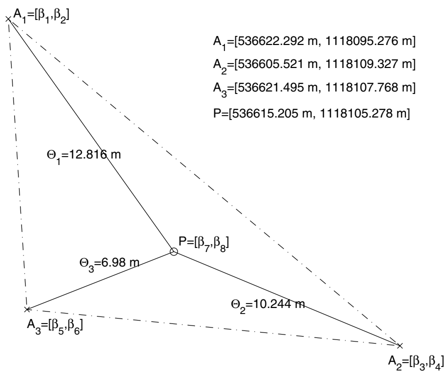


Figure 2: The polygonometric measurement.

We have given four points A_1, A_2, A_3 and P and their geographical specifications, i.e. their latitudes and longitudes, which have been obtained from a CARMIN GPS 12XL navigator. All points have been visualized on Fig. 2.

For our purposes, the geographical coordinates were transformed to the plane system known as S-JTSK (where $+x$-axes ... south, $+y$-axes ... west). For details on S-JTSK coordinates, see [5].

So we have estimated values of $A_i = (Y_{2i-1}, Y_{2i})$, $i = 1,2,3$ and measured values of $\widehat{\Theta}^I = (\widehat{\Theta}_1^I, \widehat{\Theta}_2^I, \widehat{\Theta}_3^I)'$ in the Situation I or we have estimated values of $A_i = (Y_{2i-1}, Y_{2i})$, $i = 1,2,3$, and $P = (Y_7, Y_8)$ and measured values of $\widehat{\Theta}^{II} = (\widehat{\Theta}_1^{II}, \widehat{\Theta}_2^{II}, \widehat{\Theta}_3^{II})'$ in the Situation II.

Let the result from the first and the second stage of measurement in the Situation I be $(\widehat{\Theta}_1^I, \widehat{\Theta}_2^I, \widehat{\Theta}_3^I)' = (16.683 \text{ m}, 12.453 \text{ m}, 21.613 \text{ m})'$ and

$$\mathbf{Y}^{Ig} = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \\ Y_6 \end{pmatrix} = \begin{pmatrix} 49°38'02.2'' \\ 17°23'35.1'' \\ 49°38'01.8'' \\ 17°23'36.0'' \\ 49°38'01.8'' \\ 17°23'35.2'' \end{pmatrix} \rightarrow \mathbf{Y}^I = \begin{pmatrix} 536622.292 \text{ m} \\ 1118095.276 \text{ m} \\ 536605.521 \text{ m} \\ 1118109.327 \text{ m} \\ 536621.495 \text{ m} \\ 1118107.768 \text{ m} \end{pmatrix}.$$

In the Situation II, let the result from the first and the second stage of measurement be $(\widehat{\Theta}_1^{II}, \widehat{\Theta}_2^{II}, \widehat{\Theta}_3^{II})' = (12.816 \text{ m}, 10.244 \text{ m}, 6.980 \text{ m})'$ and

$$\mathbf{Y}^{IIg} = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \\ Y_6 \\ Y_7 \\ Y_8 \end{pmatrix} = \begin{pmatrix} 49°38'02.2'' \\ 17°23'35.1'' \\ 49°38'01.8'' \\ 17°23'36.0'' \\ 49°38'01.8'' \\ 17°23'35.2'' \\ 49°38'01.9'' \\ 17°23'35.5'' \end{pmatrix} \rightarrow \mathbf{Y}^{II} = \begin{pmatrix} 536622.292 \text{ m} \\ 1118095.276 \text{ m} \\ 536605.521 \text{ m} \\ 1118109.327 \text{ m} \\ 536621.495 \text{ m} \\ 1118107.768 \text{ m} \\ 536614.788 \text{ m} \\ 1118105.885 \text{ m} \end{pmatrix}.$$

The accuracy of the measurement is given by the covariance matrix $\begin{pmatrix} \mathbf{V}_0 & \mathbf{0} \\ \mathbf{0} & \sigma^2\mathbf{V}_1 \end{pmatrix}$. Let $\widehat{\Theta}_1$, $\widehat{\Theta}_2$, $\widehat{\Theta}_3$ be the random variables with the mean values Θ_1, Θ_2, Θ_3, then

$$\mathbf{Y}_1 = \begin{pmatrix} \widehat{\Theta}_1 \\ \widehat{\Theta}_2 \\ \widehat{\Theta}_3 \end{pmatrix} \sim N_3 \left[\mathbf{X}_1 \begin{pmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \end{pmatrix}; \mathbf{V}_0 \right].$$

In our case, we will consider the covariance matrices in the form of

$$\mathbf{V}_0 = \sigma_d^2 \times \begin{pmatrix} 1, & 0, & 0 \\ 0, & 1, & 0 \\ 0, & 0, & 1 \end{pmatrix},$$

where $\sigma_d^2 = 0.01^2\text{m}^2$ and $\mathbf{X}_1 = \mathbf{I}_{3,3}$. Note that $\sigma_d^2 = 0.01^2\text{m}^2$, especially the value of 0.01 m, is usually used for the value of the standard deviation of the measuring tape.

Let $Y_1, Y_2, Y_3, Y_4, Y_5, Y_6$ be the random variables with the mean values β_1, β_2, β_3, β_4, β_5, β_6, respectively, and dispersions $\sigma^2\mathbf{V}_1$.

$$\mathbf{Y}_2 = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \\ Y_6 \end{pmatrix} \sim N_6 \left[\mathbf{X}_2 \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \end{pmatrix} ; \sigma^2\mathbf{V}_1 \right].$$

We can use the covariance matrix in the form of

$$\mathbf{\Sigma}_{22} = \sigma^2\mathbf{V}_1 = \begin{pmatrix} \cos^2\varphi, & 0, & 0, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0, & 0, & 0 \\ 0, & 0, & \cos^2\varphi, & 0, & 0, & 0 \\ 0, & 0, & 0, & 1, & 0, & 0 \\ 0, & 0, & 0, & 0, & \cos^2\varphi, & 0 \\ 0, & 0, & 0, & 0, & 0, & 1 \end{pmatrix},$$

where $\sigma^2 = 3.1^2\text{m}^2$, $\cos(\varphi) = \cos(49°) = 0.6564$ and $\mathbf{X}_2 = \mathbf{I}_{6,6}$. For the parameter σ^2 we will use the following value, calculated from

$$\sigma^2_{GPS} = \frac{2 \cdot \pi \cdot 6378 \cdot 1000}{360 \cdot 60 \cdot 60 \cdot 10} = 3.1^2\text{m}^2,$$

where the expression above, especially the value of 3.1 m, denotes the standard deviation, derived from the smallest decimal digit which the GPS reciever displays.

The angle $\varphi = 49°$ stands for the value of the latitude where the measurement has been carried out.

Finally, we have the model given by

$$\begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix} = \begin{pmatrix} \widehat{\Theta}_1 \\ \widehat{\Theta}_2 \\ \widehat{\Theta}_3 \\ Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \\ Y_6 \end{pmatrix} \sim N_9 \left[\begin{pmatrix} \mathbf{X}_1, & \mathbf{0} \\ \mathbf{0}, & \mathbf{X}_2 \end{pmatrix} \begin{pmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \end{pmatrix} ; \begin{pmatrix} \mathbf{V}_0, & \mathbf{0} \\ \mathbf{0}, & \sigma^2\mathbf{V}_1 \end{pmatrix} \right].$$

Now, we can briefly describe the core of the example. We are in the position when we have the model expressed by

$$\mathbf{Y} = \mathbf{f}(\theta) + \epsilon, \tag{1}$$

$$\text{Var}(\varepsilon) = \mathbf{\Sigma}_0, \tag{2}$$

where $\mathbf{\Sigma}_0 = \sigma^2\mathbf{V}_1 + \mathbf{V}_0$. Here we can more closely rewrite the relation (1), i.e.

$$\mathbf{Y} = \mathbf{f}(\beta) + \varepsilon = \begin{pmatrix} \sqrt{(\beta_1-\beta_3)^2+(\beta_2-\beta_4)^2} \\ \sqrt{(\beta_3-\beta_5)^2+(\beta_4-\beta_6)^2} \\ \sqrt{(\beta_1-\beta_5)^2+(\beta_2-\beta_6)^2} \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \end{pmatrix} + \varepsilon. \tag{3}$$

In our example, we will consider the covariance matrices $\mathbf{W}_0 = (0.1)^2 \times \begin{pmatrix} \mathbf{I}_{3,3}, & \mathbf{0}_{3,6} \\ \mathbf{0}_{6,3}, & \mathbf{0}_{6,6} \end{pmatrix}$ and $\sigma^2\mathbf{W}_1 = \sigma^2 \times \mathrm{diag}((0,0,0,1,\cos^2\varphi,1,\cos^2\varphi,1,\cos^2\varphi)')$ with $\sigma^2 = 3.1^2\mathrm{m}^2$ and $\cos(\varphi) = \cos(49°) = 0.6564$.

For the function $\mathbf{f}$, we will generate the Taylor expansion at the suitable point which is given by

$$\mathbf{f}(\beta^1) = \mathbf{f}(\beta^0) + \mathbf{A}(\beta^1 - \beta^0).$$

According to the theory of the measurement, we have to define the matrix $\mathbf{A}$ that is given by $\mathbf{A} = \left(\frac{\partial \mathbf{f}}{\partial \Theta'}\right)$. As an illustration, the expression for $A_{3,6}$ takes the form of $A_{3,6} = \frac{\beta_6-\beta_2}{\sqrt{\beta_5^2-2\beta_5\beta_1+\beta_1^2+\beta_6^2-2\beta_6\beta_2+\beta_2^2}}$. The derivation of the other elements, i.e., $A_{1,1}$, $A_{1,2}$, $A_{1,3}$, $A_{1,4}$, $A_{2,3}$, $A_{2,4}$, $A_{2,5}$, $A_{2,6}$, $A_{3,1}$, $A_{3,2}$ and $A_{3,5}$, is analogous.

Now we will determine the estimator $\hat{\sigma}^2$ of the parameter σ^2 according to the Theorem 2.1. The whole process of determining the estimator $\hat{\sigma}^2$ can be now, according to the Theorem 2.1, written as

$$\begin{aligned} \hat{\sigma}^2 &= \lambda\big\{[(\mathbf{Y}-\mathbf{D}\widehat{\Theta})'(\mathrm{M}_{\mathbf{A}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{A}})^+\mathbf{W}_1(\mathrm{M}_{\mathbf{A}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{A}})^+(\mathbf{Y}-\mathbf{D}\widehat{\Theta})] && (4) \\ &\quad - \mathrm{Tr}[(\mathrm{M}_{\mathbf{A}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{A}})^+\mathbf{W}_1(\mathrm{M}_{\mathbf{A}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{A}})^+\mathbf{W}_0]\big\}, && (5) \end{aligned}$$

where the value of the parameter λ is expressed by the following equation

$$\mathbf{S}_{(\mathrm{M}_{\mathbf{A}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{A}})^+}\lambda = 1, \tag{6}$$

where the $1{\times}1$ matrix $\mathbf{S}_{(\mathrm{M}_{\mathbf{A}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{A}})^+}$ takes the form of

$$\mathbf{S}_{(\mathrm{M}_{\mathbf{A}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{A}})^+} = \mathrm{Tr}[(\mathrm{M}_{\mathbf{A}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{A}})^+\mathbf{W}_1(\mathrm{M}_{\mathbf{A}}\mathbf{\Sigma}_0\mathrm{M}_{\mathbf{A}})^+\mathbf{W}_1]. \tag{7}$$

By solving equations (5),(6) and (7) we have obtained $\lambda = 4.1751e^{-27}$ and $\hat{\sigma}^2 = (0.3540\ \mathrm{m})^2$.

We can say that the estimator of the uncertainty in GPS–coordinates is $\hat{\sigma}^2 = 0.3540^2\ \mathrm{m}^2$. Hereafter, we will focus on the same model but from a different point of view. We will consider the model of the measurement (i) and condition (ii) from Definition 2.2. Finally, we have in the Situation I the model given by

$$\begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix} = \begin{pmatrix} \widehat{\Theta}_1 \\ \widehat{\Theta}_2 \\ \widehat{\Theta}_3 \\ Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \\ Y_6 \end{pmatrix} \sim N_9 \left[\begin{pmatrix} \mathbf{X}_1, & \mathbf{0} \\ \mathbf{0}, & \mathbf{X}_2 \end{pmatrix} \begin{pmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \end{pmatrix} ; \begin{pmatrix} \mathbf{\Sigma}_{11}, & \mathbf{0} \\ \mathbf{0}, & \mathbf{\Sigma}_{22} \end{pmatrix} \right].$$

In our case, $\mathbf{X}_1 = \mathbf{I}$, $\mathbf{X}_2 = \mathbf{I}$, $\mathbf{\Sigma}_{11} = (\mathbf{W}_0)_{1:3,1:3}$ and $\mathbf{\Sigma}_{22} = (\sigma^2\mathbf{W}_1)_{4:9,4:9}$ (see $\mathbf{W}_0$ and $\sigma^2\mathbf{W}_1$ on p. 168).

One can observe from Figure 2 in the Situation I that the condition $\mathbf{g}(\Theta, \beta) = \mathbf{0}$ is implied for the parameters Θ and β, where

$$\begin{aligned} g_1(\Theta, \beta) &= (\beta_5 - \beta_3)^2 + (\beta_6 - \beta_4)^2 - \Theta_1^2, \\ g_2(\Theta, \beta) &= (\beta_5 - \beta_1)^2 + (\beta_6 - \beta_2)^2 - \Theta_2^2, \\ g_3(\Theta, \beta) &= (\beta_3 - \beta_1)^2 + (\beta_4 - \beta_2)^2 - \Theta_3^2. \end{aligned}$$

The linear version of the condition $\mathbf{g}(\Theta, \beta) = \mathbf{0}$, obtained using the Taylor expansion at the approximate point $(\Theta^0, \beta^0) = (\widehat{\Theta}_1, \widehat{\Theta}_2, \widehat{\Theta}_3, Y_1, Y_2, Y_3, Y_4, Y_5, Y_6)$, is in the form of $\mathbf{B}\delta\beta + \mathbf{C}\delta\Theta + \mathbf{a} = \mathbf{0}$, where $\delta\beta = \beta - \beta^0$, $\delta\Theta = \Theta - \Theta^0$, $\mathbf{B} = \frac{\partial \mathbf{g}(\Theta^0, \beta^0)}{\partial \beta'}$, $\mathbf{C} = \frac{\partial \mathbf{g}(\Theta^0, \beta^0)}{\partial \Theta'}$ and $\mathbf{a} = \mathbf{g}(\Theta^0, \beta^0)$.

Here we present the values of the vector of the estimator $\widehat{\beta^I}$ (calculated according to Theorem 2.2) based on the model with the measurement of all triangular lengths by the measuring tape. They are as follows:

$$\widehat{\beta^I} = \begin{pmatrix} 536621.930 \text{ m} \\ 1118095.923 \text{ m} \\ 536604.643 \text{ m} \\ 1118108.123 \text{ m} \\ 536622.735 \text{ m} \\ 1118108.324 \text{ m} \end{pmatrix}.$$

Its covariance matrix was calculated (see Corollary 2.1) leading to

$$\text{Var}(\widehat{\beta^I}) = \begin{pmatrix} 1.2455 & 0.5064 & 0.0300 & -0.9437 & 0.1645 & 0.4373 \\ 0.5064 & 3.3361 & -0.2980 & 2.3743 & -0.2084 & 3.2896 \\ 0.0300 & -0.2980 & 0.7453 & 0.5556 & 0.6647 & -0.2576 \\ -0.9437 & 2.3743 & 0.5556 & 4.1672 & 0.3881 & 2.4585 \\ 0.1645 & -0.2084 & 0.6647 & 0.3881 & 0.6108 & -0.1797 \\ 0.4373 & 3.2896 & -0.2576 & 2.4585 & -0.1797 & 3.2519 \end{pmatrix}.$$

As $\mathrm{Tr}[\mathrm{Var}(\widehat{\beta}^I)] < \mathrm{Tr}(\mathbf{\Sigma}_{22})$ (see p. 169) it is evident that we get a better estimator of the coordinates of points A_1, A_2 and A_3.

Furthermore, in the same way, we will find estimator $\widehat{\beta}^{II}$ for model for the Situation II. In this situation, one can observe that the condition $\mathbf{g}(\Theta, \beta) = \mathbf{0}$ is implied for the parameters Θ and β, where

$$\begin{aligned} g_1(\Theta,\beta) &= (\beta_1 - \beta_7)^2 + (\beta_2 - \beta_8)^2 - \Theta_1^2, \\ g_2(\Theta,\beta) &= (\beta_3 - \beta_7)^2 + (\beta_4 - \beta_8)^2 - \Theta_2^2, \\ g_3(\Theta,\beta) &= (\beta_5 - \beta_7)^2 + (\beta_6 - \beta_8)^2 - \Theta_3^2. \end{aligned}$$

The linear version of the condition $\mathbf{g}(\Theta, \beta) = \mathbf{0}$, obtained using the Taylor expansion at the approximate point $(\Theta^0, \beta^0) = (\widehat{\Theta}_1, \widehat{\Theta}_2, \widehat{\Theta}_3, Y_1, Y_2, Y_3, Y_4, Y_5, Y_6)$, is in the form of $\mathbf{B}\delta\beta + \mathbf{C}\delta\Theta + \mathbf{a} = \mathbf{0}$, where $\delta\beta = \beta - \beta^0$, $\delta\Theta = \Theta - \Theta^0$, $\mathbf{B} = \frac{\partial \mathbf{g}(\Theta^0,\beta^0)}{\partial \beta'}$, $\mathbf{C} = \frac{\partial \mathbf{g}(\Theta^0,\beta^0)}{\partial \Theta'}$ and $\mathbf{a} = \mathbf{g}(\Theta^0, \beta^0)$.

Here we present the values of the vector of the estimator $\widehat{\beta}^{II}$ (calculated according to Theorem 2.2) based on the model with the measurement of 3 distances from triangular points to the inner point P by the measuring tape. The result is

$$\widehat{\beta}^{II} = \begin{pmatrix} 536622.416 \text{ m} \\ 1118094.184 \text{ m} \\ 536605.578 \text{ m} \\ 1118109.178 \text{ m} \\ 536621.752 \text{ m} \\ 1118108.403 \text{ m} \\ 536614.768 \text{ m} \\ 1118105.885 \text{ m} \end{pmatrix}.$$

Its covariance matrix (see Corollary 2.1) is given by

$$\mathrm{Var}(\widehat{\beta}^{II}) =$$
$$= \begin{pmatrix} 1.3685 & 0.6308 & 0.0797 & -0.2083 & -0.0443 & -0.1096 & 0.0361 & -0.3129 \\ 0.6308 & 3.4356 & -0.7031 & 1.8373 & 0.3907 & 0.9667 & -0.3185 & 2.7604 \\ 0.0797 & -0.7031 & 1.0072 & 1.1309 & 0.0464 & 0.1147 & 0.3067 & -0.5426 \\ -0.2083 & 1.8373 & 1.1309 & 6.0447 & -0.1212 & -0.2998 & -0.8014 & 1.4179 \\ -0.0443 & 0.3907 & 0.0464 & -0.1212 & 1.0490 & -0.9674 & 0.3889 & 0.6979 \\ -0.1096 & 0.9667 & 0.1147 & -0.2998 & -0.9674 & 6.6065 & 0.9623 & 1.7267 \\ 0.0361 & -0.3185 & 0.3067 & -0.8014 & 0.3889 & 0.9623 & 0.7083 & 0.1576 \\ -0.3129 & 2.7604 & -0.5426 & 1.4179 & 0.6979 & 1.7267 & 0.1576 & 3.0951 \end{pmatrix}.$$

As it has already been said before, we can use the outputs from the second step of our example as the input for the third part of the computation. This innovation of the algorithm could result in better estimators of our parameters.

Our task is to find the estimator of the coordinates of the point P and their covariance matrix and to determine the confidence ellipses for the coordinates of all points.

We now apply the same model like in the Situation II. We can consider another covariance matrix $\mathrm{Var}(\widehat{\beta}^I)$—the result from the Situation I, where we have better estimator of parameters than in the first stage of the measurement because of $\mathrm{Tr}[\mathrm{Var}(\widehat{\beta}^I)] < \mathrm{Tr}(\mathbf{\Sigma}_{22})$. Here we present the values of the vector of the estimator $\widehat{\beta}^{II'}$ (calculated according to Theorem 2.2). They are as follows:

$$\widehat{\beta}^{II'} = \begin{pmatrix} 536622.473 \text{ m} \\ 1118094.363 \text{ m} \\ 536605.361 \text{ m} \\ 1118109.341 \text{ m} \\ 536622.071 \text{ m} \\ 1118107.489 \text{ m} \\ 536614.607 \text{ m} \\ 1118106.183 \text{ m} \end{pmatrix}.$$

In this case, Corollary 2.1 gives the covariance matrix in the form of

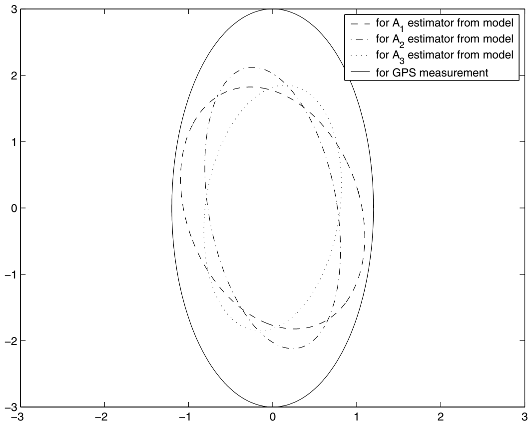
$$\mathrm{Var}(\widehat{\beta}^{II'}) =$$
$$= \begin{pmatrix} 1.1699 & 0.5461 & -0.0752 & -0.9390 & 0.0624 & 0.4752 & 0.2828 & -0.0823 \\ 0.5461 & 2.6197 & -0.2931 & 1.6152 & -0.2011 & 2.5707 & -0.0519 & 2.1943 \\ -0.0752 & -0.2931 & 0.5958 & 0.5062 & 0.5187 & -0.2560 & 0.4006 & 0.0429 \\ -0.9390 & 1.6152 & 0.5062 & 3.3453 & 0.3448 & 1.6967 & 0.0880 & 2.3429 \\ 0.0624 & -0.2011 & 0.5187 & 0.3448 & 0.4706 & -0.1745 & 0.3882 & 0.0308 \\ 0.4752 & 2.5707 & -0.2560 & 1.6967 & -0.1745 & 2.5309 & -0.0447 & 2.2017 \\ 0.2828 & -0.0519 & 0.4006 & 0.0880 & 0.3882 & -0.0447 & 0.3684 & 0.0086 \\ -0.0823 & 2.1943 & 0.0429 & 2.3429 & 0.0308 & 2.2017 & 0.0086 & 2.2611 \end{pmatrix}.$$

As we can see, it is possible to use estimator $\widehat{\beta}^I$ from the model for the Situation I for finding the estimator in the model for the Situation II.

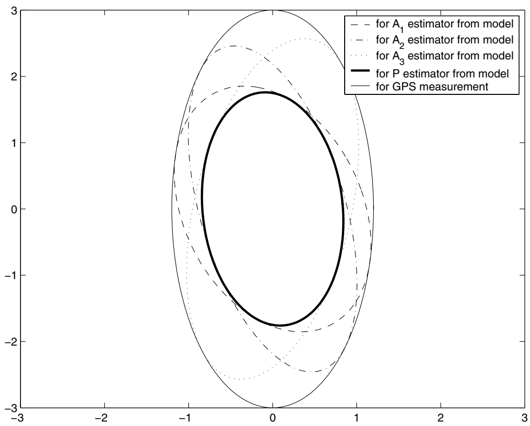
We have taken into account three different cases in which we have determined the possible way, how to obtain the coordinates from the GPS receiver, which shows a lesser uncertainty. These results, especially variances and residuals, for the first calculated situation are quite satisfactory. In the second situation we have not obtain better results because we have measured shorter distances. We have corrected this imperfection in the Situation II', where we have arrived at the best results. The essence of this method is based on the use of outputs of the Situation II as the input for the Situation II'.

The confidence ellipses obtained from calculated covariance matrix (based on Theorem 2.4) for $\alpha = 5$ % are depicted in Fig. 3.

Situation I:



Situation II:



Situation II':

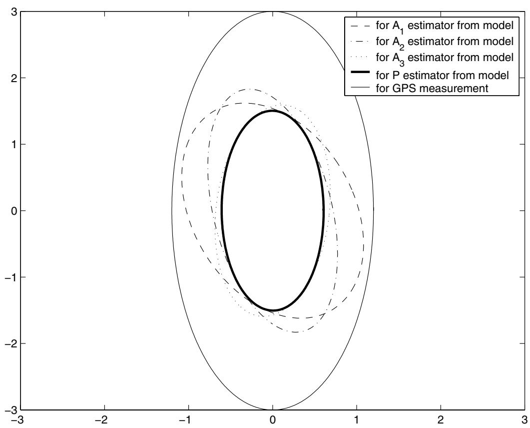


Figure 3: The (1-α) confidence ellipses for points A_1, A_2 and A_3 (solid line), for point $\hat{P}$ (bold solid line), for point $\hat{A}_1$ (dash line), for point $\hat{A}_2$ (dashdot line) and for point $\hat{A}_3$ (dot line).

4 Concluding remarks

We hope that our contribution has evidently pointed out a necessity to investigate the dispersion of the measuring device (the GPS receiver in our case) before the initiation of the measurement itself. In reality, a finding of the estimation of the dispersion can be complicated and infeasible in some cases. It may happen that the measurement cannot be repeated several times. Our proposed procedure, however, allows to estimate the dispersion without the measurement being repeated but with the help of the additional measurement (in our case, by a measure tape).

In the example worked out in this paper, we have calculated the values of the uncertainty of the GPS receiver which may have at the latitude of $\varphi = 49°$. Furthermore, our contribution have shown how the theory of estimation is a powerful tool for a modification of inaccurate data acquired by a measuring device (the GPS receiver in our case) with the utilization of the additional measurement. The example has also demonstrated a possibility of a successive improvement of the estimation by a further additional measurement.

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