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Metric of Special 2F-flat Riemannian Spaces

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Abstract

In this paper we find the metric in an explicit shape of special $2F$-flat Riemannian spaces V_n, i.e. spaces, which are $2F$-planar mapped on flat spaces. In this case it is supposed, that F is the cubic structure: $F^3 = I$.

Key words: $2F$-flat (pseudo-)Riemannian spaces, $2F$-planar mapping, cubic structure.

2000 Mathematics Subject Classification: 53B20, 53B30, 53B35

1 Introduction

$2F$- and pF-planar mappings are studied in these papers [4, 5, 17]. The mentioned mappings are the generalization of geodesic, holomorphically projective and F-planar mappings [1, 2, 6, 7, 8, 9, 10, 11, 14, 15, 16, 18].

As it is known, the Riemannian space with the constant curvature, resp. the Kählerian space with the constant holomorphically projective curvature, admits a geodesic, resp. holomorphically projective, mapping onto a flat space, i.e. the space with a vanishing curvature tensor.

The consideration in the present paper is perfomed in the tensor form, locally, in a class of substantial real smooth functions. The dimension n of the spaces under consideration, as a rule, is greater then 3. All the spaces are supposed to be connected.

We consider a (pseudo-) Riemannian space V_n with a metric tensor g and an affinor structure F, i.e. a tensor field of type $\binom{1}{1}$. We supposed, that F is the cubic affinor structure, for which it holds

$$F^3 = I.$$

In our paper we find the metric in an explicit shape of special $2F$-flat Riemannian spaces V_n, i.e. spaces, which are $2F$-planar mapped on flat spaces.

It was proved, that the Riemannian tensor of these spaces has the following form [4]:

$$R^h_{ijk} = \sum_{\sigma=0}^{2} (\overset{\sigma}{F}{}^h_i \overset{\sigma}{S}_{jk} + \overset{\sigma}{F}{}^h_j \overset{\sigma}{T}_{ik} - \overset{\sigma}{F}{}^h_k \overset{\sigma}{T}_{ij}),$$

where $\overset{\sigma}{S}_{jk}$ and $\overset{\sigma}{T}_{ik}$ are tensors. Here and after

$$\overset{0}{F}{}^h_i = \delta^h_i, \quad \overset{1}{F}{}^h_i = F^h_i, \quad \overset{2}{F}{}^h_i = F^h_\alpha F^\alpha_i,$$

where δ^h_i is the Kronecker symbol, R^h_{ijk} and F^h_i are components of the Riemannian tensor and the structure F, respectively.

Among other things it is known, that $2F$-flat Riemannian spaces V_n are symmetric, i.e. their Riemannian tensor is covariantly constant.

2 On special 2*F*-flat Rimannian space

As it was mentioned, the aim of our interest was to find the metric tensor of the $2F$-flat Riemannian spaces V_n. This problem is considerably extensive, therefore we narrow it by following assumptions.

In the following we study the $2F$-flat Riemannian spaces V_n, for which the Riemannian tensor has the form:

$$R^h_{ijk} = B\,(G^h_k G_{ij} - G^h_j G_{ik}), \tag{1}$$

where

$$G^h_k = \delta^h_i + F^h_i + F^h_\alpha F^\alpha_i, \qquad G_{ij} = g_{i\alpha} G^\alpha_j, \qquad B - \text{const.}$$

There g_{ij} are components of the metric g and F^h_i are components of the structure F, which satisfies the conditions:

$$F^3 = I, \qquad \operatorname{tr} F = \operatorname{tr} F^2 = 0, \tag{2}$$

as well the following characteristic is joined with the metric tensor:

$$\overset{1}{F}_{ij} = \overset{1}{F}_{ji} \quad \text{and} \quad \overset{2}{F}_{ij} = \overset{2}{F}_{ji}, \tag{3}$$

where $\overset{1}{F}_{ij} = g_{i\alpha} F^\alpha_j$ and $\overset{2}{F}_{ij} = g_{i\alpha} \overset{2}{F}{}^\alpha_j$.

It is clear, that V_n with this Riemannian tensor is symmetric. Therefore we use for the computation procedure of the metric tensor the formula by P. A. Shirokov [14], in accordance with this formula the metric tensor of the symmetric space in some point $M(x_0) \in V_n$ is calculate by sequences:

$$g_{ij}(y) = \underset{\circ}{g}_{ij} + \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^k \, 2^k}{(2k+2)!} \overset{(k)}{m}_{ij}, \tag{4}$$

where

$$\overset{(1)}{m}_{ij} = m_{ij}, \qquad \overset{(k+1)}{m}_{ij} = \overset{(k)}{m}_{i\alpha} m_{j\beta} \, \underset{\circ}{g}^{\alpha\beta}, \qquad m_{ij} = \underset{\circ}{R}_{i\alpha\beta j} y^\alpha y^\beta, \tag{5}$$

$\underset{\circ}{g}_{ij}$, $\underset{\circ}{g}^{ij}$, $\underset{\circ}{R}_{i\alpha\beta j}$ are values of components of the metric, inverse and Riemannian tensors in a point x_0, $y \equiv (y^1, y^2, \ \dots \ , y^n)$ are Riemannian coordinates in the point x_0.

3 The computation procedure of the metric of the 2F-flat space

We substitute (1) to (5) in some point $M(x_0)$ and obtain:

$$m_{ij} = \overset{(1)}{m}_{ij} = B\,(y_{ij} + \overset{1}{y}_{ij} + \overset{2}{y}_{ij}),$$

where

$$y_{ij} = y_i y_j + \overset{1}{y}_i \overset{2}{y}_j + \overset{2}{y}_i \overset{1}{y}_j - y \; \underset{\circ}{g}_{ij} - \overset{2}{y} \; \underset{\circ}{\overset{1}{F}}_{ij} - \overset{1}{y} \; \underset{\circ}{\overset{2}{F}}_{ij},$$

$$\overset{1}{y}_{ij} = y_{\alpha j} \; \underset{\circ}{F}{}_i^\alpha, \qquad \overset{2}{y}_{ij} = \overset{1}{y}_{\alpha j} \; \underset{\circ}{F}{}_i^\alpha,$$

$$y_i = \underset{\circ}{g}_{i\alpha} y^\alpha, \quad \overset{1}{y}_i = y_\alpha \; \underset{\circ}{F}{}_i^\alpha, \quad \overset{2}{y}_j = \overset{1}{y}_\alpha \; \underset{\circ}{F}{}_i^\alpha,$$

$$y = \underset{\circ}{g}_{\alpha\beta} y^\alpha y^\beta, \quad \overset{1}{y} = \underset{\circ}{\overset{1}{F}}_{\alpha\beta} y^\alpha y^\beta, \quad \overset{2}{y} = \underset{\circ}{\overset{2}{F}}_{\alpha\beta} y^\alpha y^\beta,$$

and $\underset{\circ}{F}{}_i^h$, $\underset{\circ}{\overset{1}{F}}{}_i^h$, $\underset{\circ}{\overset{2}{F}}{}_i^h$ are components of the corresponding tensors in the point x_0.

We notice, that

$$y_{ij} = y_{ji}, \quad \overset{1}{y}_{ij} = \overset{1}{y}_{ji}, \quad \overset{2}{y}_{ij} = \overset{2}{y}_{ji},$$

$$y_{i\alpha} \; \underset{\circ}{g}^{\alpha\beta} y_{\beta j} = -y \, y_{ij} - \overset{1}{y} \; \overset{2}{y}_{ij} - \overset{2}{y} \; \overset{1}{y}_{ij}.$$

Therefore

$$\overset{(2)}{m}_{ij} = -3B^2 (y + \overset{1}{y} + \overset{2}{y})(y_{ij} + \overset{1}{y}_{ij} + \overset{2}{y}_{ij}) = A \; \overset{(1)}{m}_{ij} = A \, m_{ij},$$

where

$$A = -3B\,(y + \overset{1}{y} + \overset{2}{y}).$$

By analogy we obtain

$$\overset{(3)}{m}_{ij} = A\,\overset{(2)}{m}_{ij} = A^2\,m_{ij},\quad \cdots,\quad \overset{(k)}{m}_{ij} = A^{k-1}\,m_{ij}.$$

Then we substitute this one to (4) and we obtain

$$g_{ij}(y) = \underset{\circ}{g}_{ij} + \frac{1}{2}\,m_{ij}\sum_{k=1}^{\infty}\frac{(-1)^k 2^k A^{k-1}}{(2k+2)!}.$$

We make sure of the convergency of the sequences for an arbitrary value of coordinates y^h.

These sequences can be introduced in the following form

$$g_{ij}(y) = \underset{\circ}{g}_{ij} + \frac{1}{4A^2}\,m_{ij}\left(1 - A - \sum_{k=0}^{\infty}\frac{(-2A)^k}{(2k)!}\right),$$

which is easy to express such as

$$g_{ij}(y) = \underset{\circ}{g}_{ij} + \frac{1}{4A^2}\,m_{ij}\left(1 - A - \left\{\begin{array}{ll}\cos\sqrt{2A}, & A > 0,\\ \mathrm{ch}\sqrt{2|A|}, & A < 0,\end{array}\right\}\right). \tag{6}$$

We can easily see that

$$\lim_{y\to 0} g_{ij}(y) = \underset{\circ}{g}_{ij}$$

and above functions $g_{ij}(y)$ are analytical onto domain.

Theorem 1 *Let V_n be a $2F$-flat Riemannian space and y its Riemannian coordinates. Suppose that the conditions (1), (2) and (3) hold. Then the metric V_n is expressed by the formula (6).*

References

[1] Beklemishev, D. V.: *Differential geometry of spaces with almost complex structure.* Geometria. Itogi Nauki i Tekhn., All-Union Inst. for Sci. and Techn. Information (VINITI), Akad. Nauk SSSR, Moscow, 1965, 165–212.

[2] Eisenhart, L. P.: Riemannian Geometry. *Princenton Univ. Press*, 1926.

[3] Kurbatova, I. N.: *HP-mappings of H-spaces.* Ukr. Geom. Sb., Kharkov, **27** (1984), 75–82.

[4] Al Lamy, R. J.: *About 2F-plane mappings of affine connection spaces.* Coll. on Diff. Geom., Eger (Hungary), 1989, 20–25.

[5] Al Lamy, R. J., Kurbatova, I. N.: *Invariant geometric objects of 2F-planar mappings of affine connection spaces and Riemannian spaces with affine structure of III order.* Dep. of UkrNIINTI (Kiev), 1990, No. 1004Uk90, 51p.

[6] Al Lamy, R. J., Mikeš, J., Škodová, M.: *On linearly pF-planar mappings.* Diff. Geom. and its Appl. Proc. Conf. Prague, 2004, Charles Univ., Prague (Czech Rep.), 2005, 347–353.

[7] J. Mikeš, *On special F-planar mappings of affine-connected spaces.* Vestn. Mosk. Univ., 1994, 3, 18–24.

[8] Mikeš, J.: *Geodesic mappings of affine-connected and Riemannian spaces.* J. Math. Sci., New York, **78**, 3 (1996), 311–333.

[9] Mikeš, J.: *Holomorphically projective mappings and their generalizations.* J. Math. Sci., New York, **89**, 3 (1998), 1334–1353.

[10] Mikeš, J., Pokorná, O.: *On holomorphically projective mappings onto Kählerian spaces.* Suppl. Rend. Circ. Mat. Palermo, II. Ser. 69, (2002), 181–186.

[11] Mikeš, J., Sinyukov, N. S.: *On quasiplanar mappings of spaces of affine connection.* Sov. Math. **27**, 1 (1983), 63–70; translation from Izv. Vyssh. Uchebn. Zaved., Mat., **248**, 1 (1983), 55–61.

[12] Petrov, A. Z.: New Method in General Relativity Theory. *Nauka, Moscow*, 1966.

[13] Petrov, A. Z.: *Simulation of physical fields.* In: Gravitation and the Theory of Relativity, Vol. 4–5, Kazan' State Univ., Kazan, 1968, 7–21.

[14] Shirokov, P. A.: Selected Work in Geometry. *Kazan State Univ. Press, Kazan*, 1966.

[15] Sinyukov, N. S.: Geodesic Mappings of Riemannian Spaces. *Nauka, Moscow*, 1979.

[16] Sinyukov, N. S.: *Almost geodesic mappings of affinely connected and Riemannian spaces.* J. Sov. Math. **25** (1984), 1235–1249.

[17] Škodová, M., Mikeš, J., Pokorná, O.: *On holomorphically projective mappings from equiaffine symmetric and recurrent spaces onto Kählerian spaces.* Circ. Mat. di Palermo, Suppl. Rend. Circ. Mat. Palermo, II. Ser. 75, (2005), 309–316.

[18] Yano, K.: Differential Geometry on Complex and Almost Complex Spaces. *Pergamon Press, Oxford–London–New York–Paris–Frankfurt*, XII, 1965, 323p.

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A Groupoid Characterization of Orthomodular Lattices *

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Abstract

We prove that an orthomodular lattice can be considered as a groupoid with a distinguished element satisfying simple identities.

Key words: Orthomodular lattice, ortholattice, orthocomplementation, OML-algebra.

2000 Mathematics Subject Classification: 06C15

A bounded lattice is called an *ortholattice* if there is a unary operation $x \mapsto x^\perp$ called *orthocomplementation* such that

$x \vee x^\perp = 1$ and $x \wedge x^\perp = 0$ (i.e. $x^\perp$ is a *complement* of x)

$x^{\perp\perp} = x$ (it is an *involution*)

$x \leq y$ implies $y^\perp \leq x^\perp$ (it is *antitone*).

An ortholattice is thus considered as an algebra $\mathcal{L} = (L; \vee, \wedge, ^\perp, 0, 1)$ of type $(2, 2, 1, 0, 0)$. Due to the above mentioned properties of orthocomplementation, it satisfies the De Morgan laws, i.e.

$(x \vee y)^\perp = x^\perp \wedge y^\perp$ and $(x \wedge y)^\perp = x^\perp \vee y^\perp$.

Hence, it can be considered also in the signature $(\vee, ^\perp, 0)$ of type $(2, 1, 0)$ because $\wedge$ can be expressed by De Morgan laws as a term function in $\vee$ and $^\perp$ and $1 = 0^\perp$.

An ortholattice $\mathcal{L} = (L; \vee, \wedge, ^\perp, 0, 1)$ is called *orthomodular* if it satisfies the implication

$x \leq y \Rightarrow x \vee (x^\perp \wedge y) = y$ (the *orthomodular law*)

which is equivalent to $x \leq y \Rightarrow y \wedge (y^\perp \vee x) = x$.

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The orthomodular law is apparently equivalent to the following identity

$$x \vee (x^{\perp} \wedge (x \vee y)) = x \vee y \qquad \text{(OMI)}$$

or, equivalently,

$$(x \vee y) \wedge ((x \vee y)^{\perp} \vee x) = x.$$

In what follows we will show that an orthomodular lattice can be discern as an algebra of type $(2, 0)$ in the signature $(\circ, 0)$, i.e. as a groupoid with a distingushed element. Let us note that Boolean algebras were characterized in this way already by the author in [4].

Definition 1 An algebra $\mathcal{A} = (A; \circ, 0)$ of type $(2, 0)$ is called an *OI-algebra* if it satisfies the following identities

(I0) $0 \circ x = 1$, where 1, denotes $0 \circ 0$

(I1) $(x \circ y) \circ x = x$

(I2) $(x \circ y) \circ y = (y \circ x) \circ x$

The proofs of the following lemmas are taken from [1].

Lemma 1 *Every OI-algebra satisfies the following identities*

(a) $x \circ (x \circ y) = x \circ y$

(b) $x \circ x = (x \circ y) \circ (x \circ y)$

Proof Applying (I1) twice, we obtain $x \circ (x \circ y) = ((x \circ y) \circ x) \circ (x \circ y) = x \circ y$, proving (a). For (b), we apply (I1), (I2) and (a):

$$x \circ x = ((x \circ y) \circ x) \circ x = (x \circ (x \circ y)) \circ (x \circ y) = (x \circ y) \circ (x \circ y). \qquad \square$$

Lemma 2 *Every OI-algebra satisfies the identities*

$$x \circ x = 1, \quad 1 \circ x = x, \quad x \circ 1 = 1.$$

Proof By Lemma 1(b) used twice we conclude $x \circ x = (x \circ y) \circ (x \circ y) = ((x \circ y) \circ y) \circ ((x \circ y) \circ y) = ((y \circ x) \circ x) \circ ((y \circ x) \circ x)(y \circ x) \circ (y \circ x) = y \circ y$. For $y = 0$ we obtain $x \circ x = 0 \circ 0 = 1$.

Now, $1 \circ x = (x \circ x) \circ x = x$ by (I1) and $x \circ 1 = x \circ (x \circ x) = x \circ x = 1$ by Lemma 1 and the firstly proved identity. $\square$

Definition 2 An OI-algebra $\mathcal{A} = (A; \circ, 0)$ is called *antitone* if it satisfies the identity

(I3) $(((x \circ y) \circ y) \circ z) \circ (x \circ z) = 1$ (where $1 = 0 \circ 0$).

Lemma 3 *Let $\mathcal{A} = (A; \circ, 0)$ be an antitone OI-algebra. Define a binary relation $\leq$ on A as follows*

$$x \leq y \quad \text{if and only if} \quad x \circ y = 1.$$

Then $\leq$ is an order on A such that $0 \leq x \leq 1$ for each $x \in A$ and

$$x \leq y \quad \text{implies} \quad y \circ z \leq x \circ z \quad \text{for all } x, y, z \in A.$$

Proof Due to Lemma 2, $\leq$ is reflexive.

Suppose $x \leq y$ and $y \leq x$. Then $x \circ y = 1$ and $y \circ x = 1$ thus, by (I2), $y = 1 \circ y = (x \circ y) \circ y = (y \circ x) \circ x = 1 \circ x = x$, i.e. $\leq$ is antisymmetric. Prove transitivity of $\leq$. Let $x \leq y$ and $y \leq z$. Then $x \circ y = 1$, $y \circ z = 1$ and, by (I3),

$$\begin{aligned} 1 &= (((x \circ y) \circ y) \circ z) \circ (x \circ z) = ((1 \circ y) \circ z) \circ (x \circ z) \\ &= (y \circ z) \circ (x \circ z) = 1 \circ (x \circ z) = x \circ z \end{aligned}$$

thus $x \leq z$. Hence, $\leq$ is an order on A. Due to (I0), $0 \leq x$ and, by Lemma 2, $x \leq 1$ for each $x \in A$.

Suppose $x \leq y$. Then $x \circ y = 1$ and, by (I3),

$$(y \circ z) \circ (x \circ z) = (((x \circ y) \circ y) \circ z) \circ (x \circ z) = 1,$$

whence $y \circ z \leq x \circ z$. □

In spite of Lemma 3, the relation $\leq$ on an antitone OI-algebra $\mathcal{A}$ will be called the *induced order of* $\mathcal{A}$.

Theorem 1 *Let $\mathcal{A} = (A; \circ, 0)$ be an antitone OI-algebra, $\leq$ the induced order on A. Then $(A; \leq)$ is a bounded lattice where $x \vee y = (x \circ y) \circ y$, and the mapping $x \mapsto x \circ 0$, is an antitone involution on $(A; \leq)$.*

Proof Since $y \leq 1$ for each $y \in A$, Lemma 3 yields $x = 1 \circ x \leq y \circ x$, i.e. $\mathcal{A}$ satisfies the identity

$$x \circ (y \circ x) = 1. \tag{B}$$

Suppose now $a, b \in A$. Then, by (B), $b \circ ((a \circ b) \circ b) = 1$ and, by (B) and (I2), $a \circ ((a \circ b) \circ b) = a \circ ((b \circ a) \circ a) = 1$, i.e. $a \leq (a \circ b) \circ b$ and $b \leq (a \circ b) \circ b$.

Suppose further $a \leq c$ and $b \leq c$. Then $b \circ c = 1$ and, by Lemma 3, $c \circ b \leq a \circ b$. Hence

$$(a \circ b) \circ b \leq (c \circ b) \circ b = (b \circ c) \circ c = 1 \circ c = c.$$

We have shown that $(a \circ b) \circ b$ is the least common upper bound of a, b, i.e.

$$a \vee b = (a \circ b) \circ b$$

and $(A; \vee)$ is a $\vee$-semilattice.

Consider the mapping $x \mapsto x \circ 0$. Then $(x \circ 0) \circ 0 = x \vee 0 = x$, i.e. it is an involution on A. By Lemma 3, this involution is antitone. Hence, we can apply De Morgan law to prove $a \wedge b = ((a \circ 0) \vee (b \circ 0)) \circ 0$ for each $a, b \in A$, i.e. $(A; \vee, \wedge)$ is a bounded lattice. □

Definition 3 An antitone OI-algebra is called an *OML-algebra* if it satisfies the identity

(I4) $(x \circ y) \circ y = (((x \circ y) \circ y) \circ 0) \circ x.$

Remark 1 By Theorem 1, (I4) can be read as

$$x \vee y = ((x \vee y) \circ 0) \circ x \tag{C}$$

which being equivalent to

$$x \leq y \Rightarrow y = (y \circ 0) \circ x. \tag{D}$$

Let $\mathcal{A}$ be an antitone OI-algebra, $\leq$ its induced order. By Theorem 1, $(A; \leq)$ is a bounded lattice. Denote this lattice by $\mathcal{L}(\mathcal{A})$ and call it the *assigned lattice of* $\mathcal{A}$.

Theorem 2 *Let $\mathcal{A} = (A; \circ, 0)$ be an OML-algebra. Then its assigned lattice $\mathcal{L}(\mathcal{A})$ is an orthomodular lattice where the orthocomplement of $x \in A$ is*

$$x^{\perp} = x \circ 0.$$

Proof Take $y = 0$ in (I4). We obtain

$$x = (x \circ 0) \circ 0 = (((x \circ 0) \circ 0) \circ 0) \circ x = (x \circ 0) \circ x,$$

thus

$$1 = x \circ x = ((x \circ 0) \circ x) \circ x = (x \circ 0) \vee x.$$

By Theorem 1, $x \mapsto x \circ 0$ is an antitone involution, thus, due to De Morgan laws,

$$0 = (x \circ 0) \wedge x$$

and hence $x^{\perp} = x \circ 0$ is an orthocomplement of $x \in A$.

By Theorem 1, we obtain immediately

$$x \circ y = ((x \circ y) \circ y) \circ y. \tag{E}$$

It remains to prove the orthomodular law. Let $x \leq y$. Then $x \circ y = 1$ and, by (I4), (I2) and (E), we derive

$$\begin{aligned} y &= (y \circ 0) \circ x = (((y \circ 0) \circ x) \circ x) \circ x = ((x \circ (y \circ 0)) \circ (y \circ 0)) \circ x \\ &= ((((x \circ (y \circ 0)) \circ (y \circ 0)) \circ 0) \circ x) \circ x = (((((y \circ 0) \circ x) \circ x) \circ 0) \circ x) \circ x \\ &= (y^{\perp} \vee x)^{\perp} \vee x = (y \wedge x^{\perp}) \vee x. \end{aligned}$$

Thus the assigned lattice $\mathcal{L}(\mathcal{A})$ is an orthomodular lattice. □

Also, conversely, to every orthomodular lattice $\mathcal{L} = (L; \vee, \wedge, ^{\perp}, 0, 1)$ an OML-algebra can be assigned as follows.

Theorem 3 *Let $\mathcal{L} = (L; \vee, \wedge, {}^\perp, 0, 1)$ be an orthomodular lattice. Consider the term function*

$$x \circ y = (x \vee y)^\perp \vee y.$$

Then $\mathcal{A}(\mathcal{L}) = (L; \circ, 0)$ is an OML-algebra.

Proof Of course, $0 \circ 0 = 0^\perp \vee 0 = 1 \vee 0 = 1$. Further,

$$0 \circ x = (0 \vee x)^\perp \vee x = x^\perp \vee x = 1$$

proving (I0). To prove (I2), we use the identity (OMI) equivalent to the orthomodular law:

$$\begin{aligned}(x \circ y) \circ y &= (((x \vee y)^\perp \vee y) \vee y)^\perp \vee y = ((x \vee y)^\perp \vee y)^\perp \vee y \\ &= ((x \vee y) \wedge y^\perp) \vee y = x \vee y,\end{aligned}$$

i.e. also $(y \circ x) \circ x = y \vee x = x \vee y = (x \circ y) \circ y$. We prove (I1):

$$(x \circ y) \circ x = (((x \vee y)^\perp \vee y) \vee x)^\perp \vee x = 1^\perp \vee x = 0 \vee x = x.$$

For (I3), we firstly prove the following

Claim: $x \leq y$ if and only if $x \circ y = 1$.

Proof: If $x \leq y$ then $x \circ y = (x \vee y)^\perp \vee y = y^\perp \vee y = 1$. Conversely, suppose $x \circ y = 1$. Then $(x \vee y)^\perp \vee y = 1$, hence by the orthomodular law

$$x \vee y = (x \vee y) \wedge ((x \vee y)^\perp \vee y) = y,$$

i.e. $x \leq y$. □

Due to the previous part and the Claim, (I3) can be rewritten as

$$(x \vee y) \circ z \leq x \circ z.$$

However,

$$(x \vee y) \circ z = (x \vee y \vee z)^\perp \vee z \leq (x \vee z)^\perp \vee z = x \circ z$$

thus (I3) is valid in $\mathcal{A}(\mathcal{L})$.

It remains to prove (I4). We have by (OMI)

$$\begin{aligned}(x \circ y) \circ y = x \vee y &= ((x \vee y) \wedge x^\perp) \vee x = ((x \vee y)^\perp \vee x)^\perp \vee x \\ &= ((x \vee y) \circ 0) \circ x = (((x \circ y) \circ y) \circ 0) \circ x.\end{aligned}$$ □

Remark 2 Since $\circ$ is a term function in $\vee$ and ${}^\perp$ and $\vee, \wedge, {}^\perp$ are term functions in $\circ$ and 0, one can easily verify that the assigning of an OML-algebra to an orthomodular lattice and conversely are mutual inverse correspondences, hence we have

$$\mathcal{L}(\mathcal{A}(\mathcal{L})) = \mathcal{L} \quad \text{and} \quad \mathcal{A}(\mathcal{L}(\mathcal{A})) = \mathcal{A}.$$

References

[1] Abbott, J. C.: *Semi-boolean algebras.* Matem. Vesnik, **4** (1967), 177–198.

[2] Birkhoff, G.: Lattice Theory. *Proc. Amer. Math. Soc., Providence, R. I.*, third edition, 1967.

[3] Beran, L.: Orthomodular Lattices – Algebraic Approach. *D. Reidel, Dordrecht*, 1984.

[4] Chajda, I.: *A groupoid characterization of Boolean algebras.* Discuss. Math., General algebra and appl., to appear.

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Sheffer Operation in Ortholattices *

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Abstract

We introduce the concept of Sheffer operation in ortholattices and, more generally, in lattices with antitone involution. By using this, all the fundamental operations of an ortholattice or a lattice with antitone involution are term functions built up from the Sheffer operation. We list axioms characterizing the Sheffer operation in these lattices.

Key words: Ortholattice, orthocomplementation, lattice with antitone involution, Sheffer operation.

2000 Mathematics Subject Classification: 06C15, 06E30

The concept of Sheffer operation (the so-called Sheffer stroke in [1]) was introduced by H. M. Sheffer in 1913. H. M. Sheffer [3] showed that all Boolean functions could be obtained from a single binary operation as term operations. In what follows, we are going to show that this works also in ortholattices and, more generally, in lattices with antitone involution and we will set up an equational axiomatization of this Sheffer operation.

Our basic concepts are taken from [1] and [2]. By a *bounded lattice* we mean a lattice with least element $\mathbf{0}$ and greatest element $\mathbf{1}$. Let $\mathcal{L} = (L; \vee, \wedge)$ be a lattice. A mapping $x \mapsto x^\perp$ is called an antitone involution on $\mathcal{L}$ if

$x \leq y$ implies $y^\perp \leq x^\perp$ (antitone)

$x^{\perp\perp} = x$ (involution).

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The fact that an antitone involution $^\perp$ is a unary operation of $\mathcal{L}$ will be expressed by the notation $\mathcal{L} = (L; \vee, \wedge, ^\perp)$. If $\mathcal{L}$ is a bounded lattice with an antitone involution which is, moreover, a complementation on $\mathcal{L}$, i.e. it satisfies

$$x \vee x^\perp = \mathbf{1} \quad \text{and} \quad x \wedge x^\perp = \mathbf{0},$$

then $x^\perp$ is called an *orthocomplement* of x and $\mathcal{L} = (L; \vee, \wedge, ^\perp, \mathbf{0}, \mathbf{1})$ an *ortholattice*.

It is worth noticing that if $^\perp$ is an antitone involution on $\mathcal{L}$, then $\mathcal{L} = (L; \vee, \wedge, ^\perp)$ satisfies the *De Morgan laws*

$$x^\perp \vee y^\perp = (x \wedge y)^\perp \quad \text{and} \quad x^\perp \wedge y^\perp = (x \vee y)^\perp.$$

Our basic concept is the following.

Definition 1 Let $\mathcal{A} = (A; \circ)$ be a groupoid. The operation $\circ$ is called *Sheffer operation* if it satisfies the following identities:

(S1) $x \circ y = y \circ x$ (commutativity)

(S2) $(x \circ x) \circ (x \circ y) = x$ (absorption)

(S3) $x \circ ((y \circ z) \circ (y \circ z)) = ((x \circ y) \circ (x \circ y)) \circ z$

(S4) $(x \circ ((x \circ x) \circ (y \circ y))) \circ (x \circ ((x \circ x) \circ (y \circ y))) = x$ (absorption)

If, moreover, it satisfies

(S5) $y \circ (x \circ (x \circ x)) = y \circ y$,

it is called an *ortho-Sheffer operation*.

Remark 1 (S2) implies also weak idempotency $(x \circ x) \circ (x \circ x) = x$.

Lemma 1 *Let $\mathcal{A} = (A; \circ)$ be a groupoid with a Sheffer operation. Define a binary relation $\leq$ on A as follows*

$$x \leq y \quad \textit{if and only if} \quad x \circ y = x \circ x.$$

Then $\leq$ is an order on A.

Proof Reflexivity of $\leq$ is evident. Suppose $x \leq y$ and $y \leq x$. Then $x \circ y = x \circ x$ and $x \circ y = y \circ x = y \circ y$, i.e. $x \circ x = y \circ y$ and hence by (S2) also $x = (x \circ x) \circ (x \circ x) = (y \circ y) \circ (y \circ y) = y$. Thus $\leq$ is antisymmetric. Suppose $x \leq y$ and $y \leq z$. Then $x \circ y = x \circ x$, $y \circ z = y \circ y$ and hence $(x \circ y) \circ (x \circ y) = (x \circ x) \circ (x \circ x) = x$, i.e. by (S3) and (S2) also

$$\begin{aligned}(x \circ z) &= ((x \circ y) \circ (x \circ y)) \circ z = x \circ ((y \circ z) \circ (y \circ z)) \\ &= x \circ ((y \circ y) \circ (y \circ y)) = x \circ y = x \circ x\end{aligned}$$

proving $x \leq z$. Thus $\leq$ is also transitive and hence it is an order on A. □

Because of Lemma 1, $\leq$ will be called the *induced order* of $\mathcal{A} = (A; \circ)$.

Lemma 2 *Let $\circ$ be a Sheffer operation on A and $\leq$ the induced order of $\mathcal{A} = (A;\circ)$. Then*

(a) $x \leq y$ *if and only if* $y \circ y \leq x \circ x$*;*

(b) $x \circ (y \circ (x \circ x)) = x \circ x$ *is the identity of* $\mathcal{A}$*;*

(c) $x \leq y$ *implies* $y \circ z \leq x \circ z$*;*

(d) $a \leq x$ *and* $a \leq y$ *imply* $x \circ y \leq a \circ a$*.*

Proof (a) If $x \leq y$ then $x \circ y = x \circ x$ and, by (S2),

$$(x \circ x) \circ (y \circ y) = (x \circ y) \circ (y \circ y) = y = (y \circ y) \circ (y \circ y)$$

thus $y \circ y \leq x \circ x$.

Conversely, if $y \circ y \leq x \circ x$ then, analogously, we can prove

$$(x \circ x) \circ (x \circ x) \leq (y \circ y) \circ (y \circ y)$$

which, by (S2), yields $x \leq y$.

(b) This identity follows directly by (S2) if $x \circ x$ is considered instead of x:

$$x \circ x = ((x \circ x) \circ (x \circ x)) \circ ((x \circ x) \circ y) = x \circ ((x \circ x) \circ y) = x \circ (y \circ (x \circ x)).$$

(c) Let $x \leq y$. Then $x \circ y = x \circ x$, i.e.

$$(x \circ y) \circ (x \circ y) = (x \circ x) \circ (x \circ x) = x$$

and hence, by (S3),

$$(y \circ z) \circ (x \circ z) = (y \circ z) \circ (((x \circ y) \circ (x \circ y)) \circ z)$$
$$(y \circ z) \circ (x \circ ((y \circ z) \circ (y \circ z))) = (y \circ z) \circ (y \circ z)$$

by the previous identity (b). Thus $y \circ z \leq x \circ z$.

(d) Suppose $a \leq x$ and $a \leq y$. Then by (c),

$$a \circ a \geq x \circ a \quad \text{and} \quad x \circ a = a \circ x \geq y \circ x.$$

Using transitivity of $\leq$, we conclude $a \circ a \geq y \circ x = x \circ y$. □

Theorem 1 *Let $\circ$ be a Sheffer operation on A and $\leq$ the induced order on $\mathcal{A} = (A, \circ)$. Define*

$$x \vee y = (x \circ x) \circ (y \circ y), \quad x^{\perp} = x \circ x \quad \text{and} \quad x \wedge y = (x^{\perp} \vee y^{\perp})^{\perp}.$$

Then $\mathcal{L}(\mathcal{A}) = (A; \vee, \wedge, {}^{\perp})$ is a lattice with antitone involution.

Proof By (S2) and (S4) we obtain

$$x \circ ((x \circ x) \circ (y \circ y)) = ((x \circ ((x \circ x) \circ (y \circ y))) \circ (x \circ ((x \circ x) \circ (y \circ y))))$$
$$\circ ((x \circ ((x \circ x) \circ (y \circ y))) \circ (x \circ ((x \circ x) \circ (y \circ y)))) = x \circ x$$

and, analogously $y \circ ((x \circ x) \circ (y \circ y)) = y \circ y$, thus $x \leq (x \circ x) \circ (y \circ y)$ and $y \leq (x \circ x) \circ (y \circ y)$. Suppose now $x \leq c$ and $y \leq c$. Then $x \circ c = x \circ x$, $y \circ c = y \circ y$ and, by Lemma 2 (c) and (d), $c = (c \circ c) \circ (c \circ c) \geq (x \circ c) \circ (y \circ c) = (x \circ x) \circ (y \circ y)$. Hence, $(x \circ x) \circ (y \circ y)$ is the least common upper bound of x, y, i.e. $x \vee y = (x \circ x) \circ (y \circ y)$.

By (S2), $x^{\perp\perp} = (x \circ x) \circ (x \circ x) = x$ and, by Lemma 2(c), the mapping $x \mapsto x^{\perp} = x \circ x$ is antitone, i.e. it is an antitone involution on $(A, \leq)$. Applying the De Morgan laws we conclude $x \wedge y = (x^{\perp} \vee y^{\perp})^{\perp}$. Hence, $\mathcal{L}(\mathcal{A}) = (A; \vee, \wedge, {}^{\perp})$ is a lattice with antitone involution. □

Because of Theorem 1, we call $\mathcal{L}(\mathcal{A})$ the *induced lattice* of $\mathcal{A} = (A, \circ)$.

Theorem 2 *Let $\circ$ be an ortho-Sheffer operation on A and $\mathcal{L}(\mathcal{A}) = (A; \vee, \wedge, {}^{\perp})$ the induced lattice. Then $\mathcal{L}(\mathcal{A})$ is an ortholattice $(A; \vee, \wedge, {}^{\perp}, \mathbf{0}, \mathbf{1})$ where $\mathbf{1} = x \circ (x \circ x)$ and $\mathbf{0} = \mathbf{1} \circ \mathbf{1}$.*

Proof Due to Theorem 1, we only need to verify that $x \circ (x \circ x)$ is the greatest element $\mathbf{1}$ of $(A, \leq)$, $\mathbf{0} = \mathbf{1} \circ \mathbf{1}$ is the least element of $(A; \leq)$ and $x^{\perp}$ is a complement of x.

By (S5) we obtain immediately $y \leq x \circ (x \circ x)$ for all $x, y \in A$. Hence $x \circ (x \circ x) = z \circ (z \circ z)$ for all $x, z \in A$, i.e. it is a constant of $(A, \circ)$ which is greater than each element $y \in A$. Denote this constant by $\mathbf{1}$. Hence, $\mathbf{0} = \mathbf{1} \circ \mathbf{1}$ is an algebraic constant of $(A; \circ)$ and, due to Lemma 2(a), $\mathbf{0} = \mathbf{1} \circ \mathbf{1} \leq y \circ y$. Taking $y = x \circ x$, we have $\mathbf{0} \leq (x \circ x) \circ (x \circ x) = x$ for each $x \in A$, i.e. $\mathbf{0}$ is the least element of $(A; \leq)$.

Applying the operations $\vee, \wedge, {}^{\perp}$ introduced in Theorem 1 we have immediately

$$x^{\perp} \vee x = ((x \circ x) \circ (x \circ x)) \circ (x \circ x) = x \circ (x \circ x) = \mathbf{1}.$$

By the De Morgan law also $x \wedge x^{\perp} = \mathbf{0}$, i.e. $x^{\perp}$ is a complement and hence an orthocomplement of x. □

Theorem 3 *Let $\mathcal{L} = (L; \vee, \wedge, {}^{\perp})$ be a lattice with antitone involution. Define*

$$x \circ y = x^{\perp} \vee y^{\perp}.$$

Then $\circ$ is Sheffer operation on L. If $\mathcal{L} = (L; \vee, \wedge, {}^{\perp}, \mathbf{0}, \mathbf{1})$ is an ortholattice then this Sheffer operation $\circ$ satisfies also (S5).

Proof (S1) is evident. We prove (S2):

$$x = x \vee (x \wedge y) = x \vee (x^{\perp} \vee y^{\perp})^{\perp} = (x \circ x) \circ (x \circ y).$$

For (S3) we compute

$$\begin{aligned} x \circ ((y \circ z) \circ (y \circ z)) &= x^{\perp} \vee (y^{\perp} \vee z^{\perp})^{\perp\perp} = x^{\perp} \vee (y^{\perp} \vee z^{\perp}) = (x^{\perp} \vee y^{\perp}) \vee z^{\perp} \\ &= (x^{\perp} \vee y^{\perp})^{\perp\perp} \vee z^{\perp} = ((x \circ y) \circ (x \circ y)) \circ z. \end{aligned}$$

We prove (S4):

$$(x \circ ((x \circ x) \circ (y \circ y))) \circ (x \circ ((x \circ x) \circ (y \circ y))) = (x^{\perp} \vee (x \vee y)^{\perp})^{\perp}$$
$$= x \wedge (x \vee y) = x.$$

Suppose now that $x^{\perp}$ is an orthocomplement of x, then

$$y \circ (x \circ (x \circ x)) = y^{\perp} \vee (x^{\perp} \vee x)^{\perp} = y^{\perp} \vee (x \wedge x^{\perp}) = y^{\perp} \vee \mathbf{0} = y^{\perp} = y \circ y$$

thus $\circ$ satisfies also (S5). □

Let $\mathcal{A} = (A; \circ)$ be a groupoid with ortho-Sheffer operation. We denoted by $\mathcal{L}(\mathcal{A})$ the ortholattice induced by $\mathcal{A}$ as considered in Theorems 1 and 2. Analogously, when given an ortholattice $\mathcal{L} = (L; \vee, \wedge, {}^{\perp}, \mathbf{0}, \mathbf{1})$ denote by $\mathcal{A}(\mathcal{L})$ the groupoid $(L, \circ)$ where $\circ$ is the ortho-Sheffer operation defined as in Theorem 3. Using Theorems 1, 2, 3 and easy computations, one can prove the following correspondence theorem.

Theorem 4 *Let* $\mathcal{L} = (L; \vee, \wedge, {}^{\perp}, \mathbf{0}, \mathbf{1})$ *be an ortholattice and* $\mathcal{A} = (A, \circ)$ *a groupoid with ortho-Sheffer operation. Then*

$$\mathcal{A}(\mathcal{L}(\mathcal{A})) = \mathcal{A} \quad \textit{and} \quad \mathcal{L}(\mathcal{A}(\mathcal{L})) = \mathcal{L}.$$

Proof The proof is an easy exercise left to the reader. □

References

[1] Birkhoff, G.: Lattice Theory. *Proc. Amer. Math. Soc., Providence, R. I.*, third edition, 1967.

[2] Grätzer, G.: General Lattice Theory. *Birkhäuser Verlag, Basel*, second edition, 1998.

[3] Sheffer, H. M.: *A set of five independent postulates for Boolean algebras.* Trans. Amer. Math. Soc. **14** (1913), 481–488.

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The Converse of Kelly's Lemma and Control-classes in Graph Reconstruction

To Professor Adriano Barlotti on the occasion of his 80th birthday

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Abstract

We prove a converse of the well-known Kelly's Lemma. This motivates the introduction of the general notions of $\mathcal{K}$-table, $\mathcal{K}$-congruence and control-class.

Key words: Graph; Kelly's Lemma; Reconstruction.

2000 Mathematics Subject Classification: 05C60

1 Introduction

An *Ulam-subgraph* of a (finite, simple, undirected, labelled) graph G of order n is a subgraph of order $n-1$ obtained from G by deleting a vertex of G and the edges incident to it. Such a subgraph can also be defined as a maximal induced subgraph of G or, simply, as a subgraph induced by $n-1$ vertices of G.

Thus, a graph G of order n gives rise to n distinct Ulam-subgraphs, the set of which is sometimes called the Ulam-deck of G. We shall denote by $G^{(v)}$ the Ulam-subgraph of G obtained by deleting the vertex v of G. Note that distinct Ulam-subgraphs may be isomorphic.

We say that two graphs X, Y *have the same Ulam-deck* if there is a one-to-one correspondence between the Ulam-decks of X and Y, such that corresponding subgraphs are isomorphic.

It is clear that two isomorphic graphs must have the same Ulam-deck. Ulam and Kelly, in 1941, have conjectured that having the same deck is a *sufficient* condition for isomorphism for all graphs of order $n \geq 3$.

Since that time, the conjecture has been verified for X, Y belonging to several classes of graphs and many other related problems have been considered (fairly recent surveys are [2] and [9]).

In Section 2, for the benefit of a reader not too familiar with Reconstruction Theory, we make a few remarks explaining (and improving) current terminology.

In Section 3 we state without proof Kelly's Lemma and one of its well-known generalizations due to Greenwell and Hemminger.

In Section 4 we prove the converse of Kelly's Lemma, a result which—although fairly easy to establish—does not seem to appear in the literature.

In Section 5 we define, for a given class $\mathcal{K}$ of graphs, the notions of $\mathcal{K}$-*table* of a graph G, of $\mathcal{K}$-*homogeneous* graphs, and of $\mathcal{K}$-*congruent* graphs. These notions suggest that we call a class $\mathcal{K}$ an *overall* (resp. *pointwise*) control-class, if two graphs X, Y are isomorphic whenever they are $\mathcal{K}$-homogeneous (resp. $\mathcal{K}$-congruent). We point out that the class of paths is not a pointwise control-class for the trees, and suggest a few classes that might be.

In Section 6 we discuss a possible strengthening of the Kelly–Ulam's conjecture.

2 Remarks on terminology and subproblems

The Kelly–Ulam's Conjecture is stated for two arbitrary given graphs X, Y: if X and Y have the same Ulam-deck, then they should be isomorphic.

In order to obtain partial results, the general problem has been split into subproblems or confined to subclasses of the class of all graphs. The following terminology has been introduced.

First of all, a graph X is called *reconstructible* if any graph Y having the same Ulam-deck as X is isomorphic to X.

Thus, proving the Kelly–Ulam's conjecture is the same as proving that any graph of order ≥ 3 is reconstructible, and partial results regarding the Kelly–Ulam's conjecture may consist in proving that restricted types of graphs (or even interesting individual graphs) are reconstructible. For example, it is easy to prove that a regular graph is reconstructible.

Another useful definition, which generalizes the one given above, is the following.

Definition 1 A graph X is *reconstructible within the class of graphs* $\mathcal{A}$ (containing X), if any graph $Y \in \mathcal{A}$, having the same Ulam-deck as X, is isomorphic to X.

Hence, the task of proving that a given graph X is reconstructible may be split into the following two steps, with respect to a suitably chosen class $\mathcal{A}$ (containing X):

- prove that X is reconstructible within $\mathcal{A}$.
- prove that an arbitrary graph Y, having the same Ulam-deck as X, must also belong to $\mathcal{A}$.

The former step is sometimes called *weak-reconstructability* of X, but we prefer to call it reconstructability within $\mathcal{A}$, and the latter *recognizability* of the class $\mathcal{A}$. If $\mathcal{A}$ is characterized by a property P, one also speaks of *recognizability* of P. Thus, the difficulty in establishing the recognizability of a class $\mathcal{A}$ strongly depends on the features of $\mathcal{A}$ and, presumably, it is greater when $\mathcal{A}$ is small. For example, it is not known whether the class of planar graphs is recognizable.

Note that if $\mathcal{A}$ is the class of all graphs isomorphic to a given X, then the recognizability of $\mathcal{A}$ is equivalent to the reconstructability of X.

3 Kelly's Lemma and its generalizations

We first introduce some notation and terminology. If X and Y have the same Ulam-deck, then, by definition, there is a one-to-one correspondence σ between the set of the Ulam-subgraphs of X and the set of the Ulam-subgraphs of Y such that corresponding Ulam-subgraphs are isomorphic. Since an Ulam-subgraph contains all but one vertex, then the one-to-one correspondence σ naturally induces another one-to-one correspondence: the correspondence π between the missing vertices. Thus, we can say that X, Y have the same Ulam-deck if and only if there is a bijection $\pi : V(X) \to V(Y)$ such that $X^{(v)} \simeq Y^{(\pi(v))}$ for all $v \in V(X)$. The bijection π will be referred to as an *Ulam-congruence*, and X will be said *Ulam-congruent* to Y.

Let Z be a graph, $v \in V(Z)$. For any graph Q, we set

$\binom{Z}{Q}$ = number of subgraphs of Z isomorphic to Q,

$\binom{Z}{Q}_v$ = number of subgraphs of Z containing vertex v isomorphic to Q.

The so-called Kelly's Lemma is the first result regarding the Kelly-Ulam's conjecture that have been obtained (in [7]). It points out a consequence of the hypothesis that two graphs are Ulam-congruent, quite remarkable in spite of the simplicity of the proof.

Lemma 1 (Kelly's Lemma) *Let X, Y be graphs of order n. Assume that there is a bijection $\pi : V(X) \to V(Y)$ such that*

(i) $X^{(v)} \simeq Y^{(\pi(v))}$ *for all $v \in V(X)$.*

Then

(ii) $\binom{X}{Q} = \binom{Y}{Q}$ *for all graphs Q of order less than n.*

(iii) $\binom{X}{Q}_v = \binom{Y}{Q}_{\pi(v)}$ *for all $v \in V(X)$ and all graphs Q of order less than n.*

We now record a generalization of Kelly's Lemma due to Greenwell and Hemminger ([5]). Let $\mathcal{F}$ be a class of graphs. An *$\mathcal{F}$-subgraph* of a graph G is a subgraph of G isomorphic to some element of $\mathcal{F}$.

Lemma 2 (Greenwell–Hemminger's Lemma) *Let $\mathcal{F}$ be a class of graphs. Let X, Y be Ulam-congruent graphs of order n. Assume that all $\mathcal{F}$-subgraphs of X and Y have order less than n, and that the intersection of two distinct maximal $\mathcal{F}$-subgraphs of X (and Y) is not an $\mathcal{F}$-subgraph. Then, for every $Q \in \mathcal{F}$, the number of maximal $\mathcal{F}$-subgraphs of X isomorphic to Q is equal to the number of maximal $\mathcal{F}$-subgraphs of Y isomorphic to Q.*

Remark 1 When $\mathcal{F}$ consists of a single graph, then the Greenweel–Hemminger's Lemma reduces to Kelly's Lemma. Also, when $\mathcal{F}$ is the set of all subgraphs of X of order exactly $n-1$, the assumption and the conclusion coincide.

Example 1 (Greenwell and Hemminger) Let X, Y and Q be as in Fig. 1.

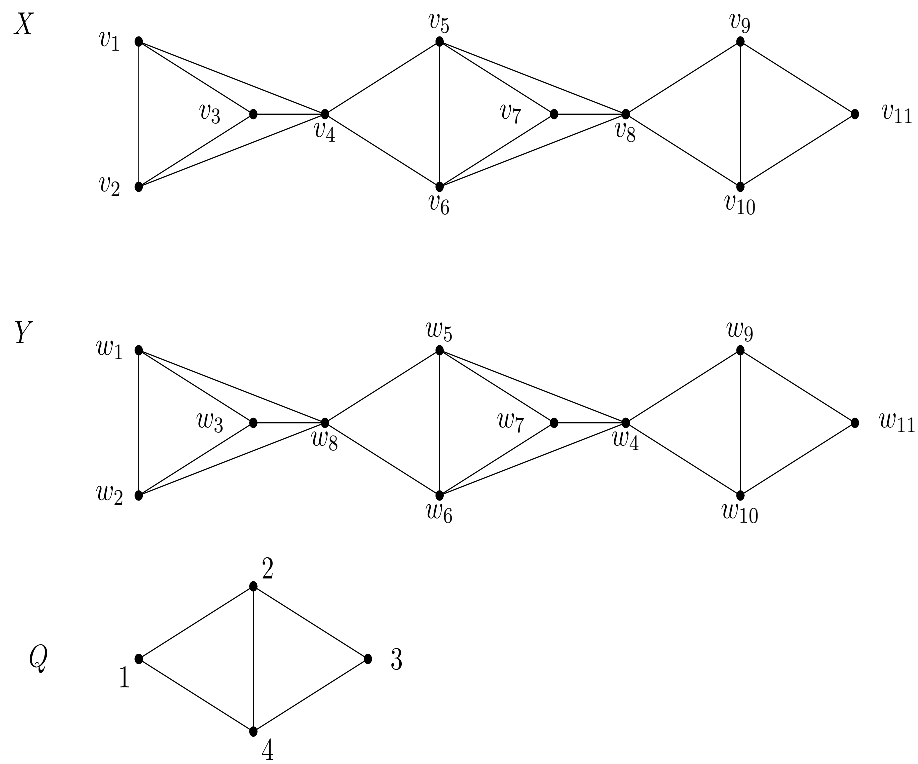

Figure 1: Example of the Greenweel–Hemminger's Lemma.

Let $\pi : v_i \to w_i$ for all i. Let $\mathcal{F}$ be the class of all 2-connected graphs. Then π is an Ulam-congruence from X to Y. The assumptions of the Greenwell–

Hemminger's Lemma are verified since the intersection of two maximal 2-connected subgraphs is not 2-connected. In both X and Y the total number of 2-con–nected maximal subgraphs isomorphic to Q equals 1. Also, there are 4 subgraphs of X isomorphic to Q containing v_4: These are

$$H_1 = \{v_1v_4, v_2v_4, v_1v_3, v_2v_3, v_3v_4\}, \quad H_2 = \{v_1v_2, v_1v_3, v_2v_3, v_1v_4, v_2v_4\},$$
$$H_3 = \{v_4v_5, v_4v_6, v_5v_7, v_5v_6, v_6v_7\}, \quad H_4 = \{v_4v_5, v_4v_6, v_5v_6, v_5v_8, v_6v_8\}.$$

For $i = 1, 2, 3, 4$, no H_i is a maximal $\mathcal{F}$-subgraph of X, i.e. it is a subgraph of X which can be properly extended to a 2-connected subgraph of X.

There are 4 subgraphs of Y isomorphic to Q containing $w_4 = \pi(v_4)$: These are

$$K_1 = \{w_5w_6, w_5w_4, w_6w_4, w_5w_7, w_6w_7\},$$
$$K_2 = \{w_5w_4, w_5w_7, w_4w_6, w_6w_7, w_4w_7\},$$
$$K_3 = \{w_4w_5, w_4w_6, w_5w_6, w_5w_8, w_6w_8\},$$
$$K_4 = \{w_4w_9, w_4w_{10}, w_9w_{10}, w_9w_{11}, w_{10}w_{11}\}.$$

Note that K_4 is a maximal $\mathcal{F}$-subgraph of Y. Thus, this example shows that a pointwise version of Greenwell–Hemminger's Lemma does not hold.

Another generalization of Kelly's Lemma is given by Tutte ([12])

4 The Converse of Kelly's Lemma

Recall that a *class* of graphs is a family of graphs closed under isomorphisms. We denote by $\mathcal{G}$ the class of all graphs. In the next theorem we collect the statement of both Kelly's Lemma and its converse and give a complete proof. Regarding the proof of the implication (ii) $\Rightarrow$ (i) (the converse of Kelly's Lemma), the reader may keep in mind the following example, where we have shown how the Ulam-subgraphs are "distributed" among the subgraphs of order $n-1$ of the various *sizes*, i.e. number of edges (see Fig. 2).

Theorem 1 *Let X, Y be graphs of order n and let $\mathcal{K}$ be a class of graphs. Let π be a bijection $V(X) \to V(Y)$. Consider the following conditions:*

(i) $X^{(v)} \simeq Y^{(\pi(v))}$ *for all* $v \in V(X)$.

(ii) $\binom{X}{Q} = \binom{Y}{Q}$ *for all* $Q \in \mathcal{K}$ *of order less than* n.

(iii) $\binom{X}{Q}_v = \binom{Y}{Q}_{\pi(v)}$ *for all* $v \in V(X)$ *and all* $Q \in \mathcal{K}$ *of order less than* n.

Then (i) $\Rightarrow$ (ii), (i) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (ii). If $\mathcal{K} = \mathcal{G}$, the three conditions are equivalent.

Proof Let $\binom{Z}{Q}_{v'}$ = number of subgraphs of Z *not* containing the vertex v and isomorphic to Q, and consider the auxiliary condition

(i)$'$ There is a bijection $\pi : V(X) \to V(Y)$ such that $\binom{X}{Q}_{v'} = \binom{Y}{Q}_{\pi(v)'}$ for all $v \in V(X)$ and all graphs $Q \in \mathcal{K}$ of order less than n.

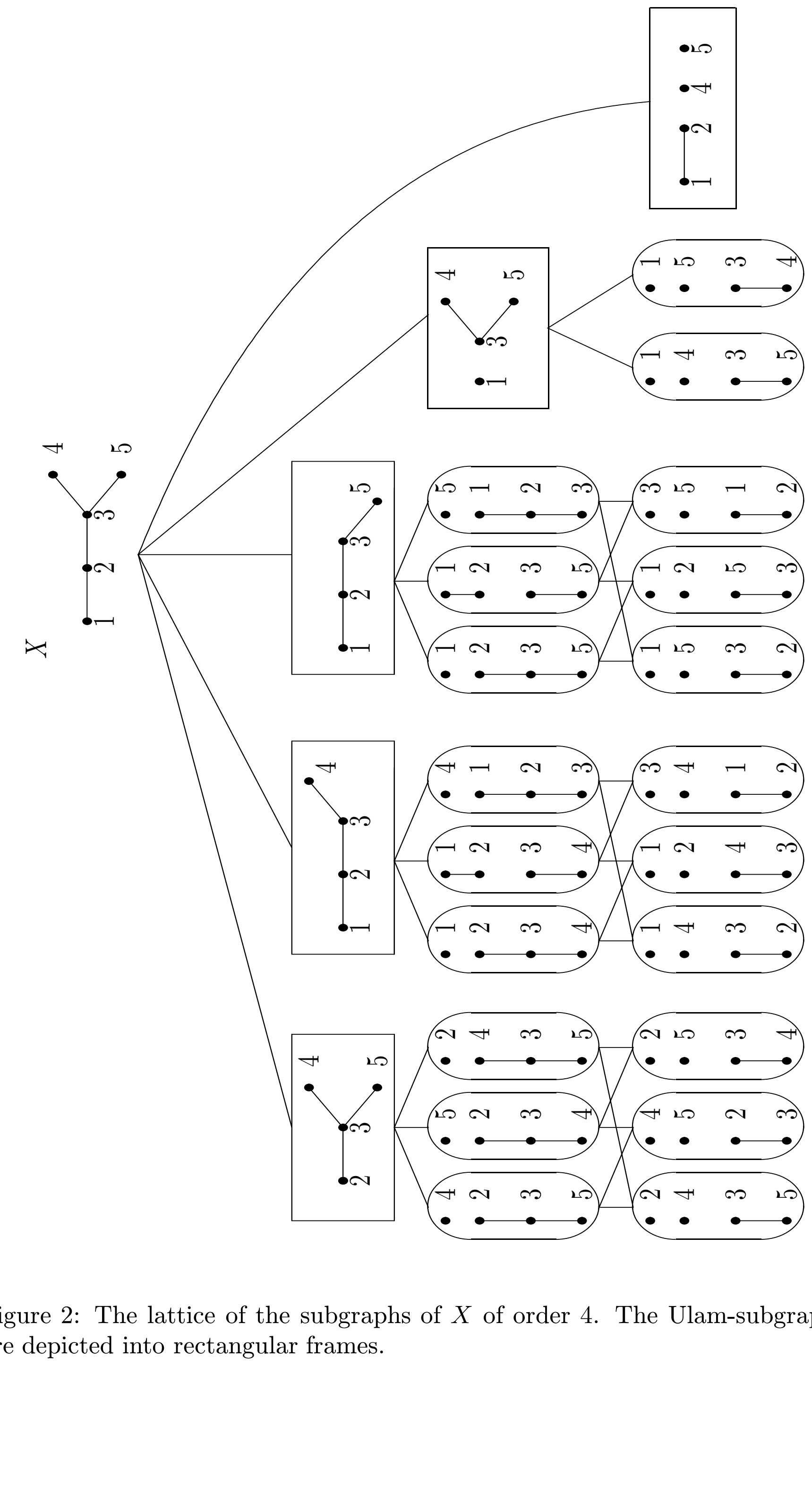

Figure 2: The lattice of the subgraphs of X of order 4. The Ulam-subgraphs are depicted into rectangular frames.

We prove that (i) $\Rightarrow$ (i)$'$ for all $\mathcal{K}$, and (i)$'$ $\Rightarrow$ (i) for $\mathcal{K} = \mathcal{G}$.

Proof of (i)$\Rightarrow$(i)$'$. By (i), $X^{(v)} \simeq Y^{(\pi(v))}$, hence $\binom{X^{(v)}}{Q} = \binom{Y^{(\pi(v))}}{Q}$ for all $Q \in \mathcal{K}$. But since $|Q| < n$, $\binom{X}{Q}_{v'} = \binom{X^{(v)}}{Q}$, and $\binom{Y}{Q}_{\pi(v)'} = \binom{Y^{(\pi(v))}}{Q}$ for all $v \in V(X)$.

Proof of (i)$'$$\Rightarrow$(i). Fix any $v \in V(X)$. Assume (i)$'$ for $\mathcal{K} = \mathcal{G}$. Thus we can replace $X^{(v)}$ for Q, thus obtaining

$$1 = \binom{X^{(v)}}{X^{(v)}} = \binom{X}{X^{(v)}}_{v'} = \binom{Y}{X^{(v)}}_{\pi(v)'} = \binom{Y^{\pi(v)}}{X^{(v)}}.$$

We can also replace $Y^{\pi(v)}$ for Q, obtaining

$$\binom{X^{(v)}}{Y^{(\pi(v))}} = \binom{X}{Y^{(\pi(v))}}_{v'} = \binom{Y}{Y^{(\pi(v))}}_{\pi(v)'} = \binom{Y^{(\pi(v))}}{Y^{(\pi(v))}} = 1.$$

In particular, we get $X^{(v)} \lesssim Y^{\pi(v)}$ from the first equality, and $Y^{(\pi(v))} \lesssim X^{(v)}$ from the second. Hence $Y^{(\pi(v))} \simeq X^{(v)}$.

Note that just one of the inequalities above, together with the finiteness of the graphs involved, would suffice to obtain the same conclusion if one proves that $X^{(v)}$ and $Y^{\pi(v)}$ have the same number of edges.

Proof of (i) $\Rightarrow$ (ii).

$$\binom{X}{Q} = \frac{1}{n - |Q|} \sum_{v \in V(X)} \binom{X^{(v)}}{Q} = \frac{1}{n - |Q|} \sum_{v \in V(X)} \binom{Y^{(\pi(v))}}{Q} = \binom{Y}{Q}.$$

Proof of (i) $\Rightarrow$ (iii). From what above, we can show that (i)$'$$\wedge$ (ii) $\Rightarrow$ (iii). Simply write

$$\binom{X}{Q}_v = \binom{X}{Q} - \binom{X}{Q}_{v'} = \binom{Y}{Q} - \binom{Y}{Q}_{\pi(v)'} = \binom{Y}{Q}_{\pi(v)}.$$

Proof of (iii) $\Rightarrow$ (ii). One can briefly argue as follows.

If, for each $v \in V(X)$, one counts the number of subgraphs of X containing v and isomorphic to (the given fixed) Q and then sums up the values obtained for various v, one overcounts each such subgraph H (isomorphic to Q) by a factor $|H|$, because for all vertices v of H the same H is counted.

As the subgraphs considered are all isomorphic to Q, they all have the same order $|Q|$. This allows us to obtain $\binom{X}{Q}$ by dividing out by $|Q|$. Then

$$\binom{X}{Q} = \frac{1}{|Q|} \sum_{v \in V(X)} \binom{X}{Q}_v = \frac{1}{|Q|} \sum_{v \in V(Y)} \binom{Y}{Q}_{\pi(v)} = \binom{Y}{Q},$$

which proves (ii).

Proof of (ii) $\Rightarrow$ (i) when $\mathcal{K} = \mathcal{G}$. One has to take into account the fact that the various Ulam-subgraphs may have different sizes (number of edges).

We shall prove (i) by assuming only that $\binom{X}{Q} = \binom{Y}{Q}$ for all graphs Q of order exactly $n-1$. In fact, in this part of the proof, all the subgraphs of X and Y considered will be subgraphs of order $n-1$.

If Q is a graph of order $n-1$, denote by $\mathcal{U}_X(Q)$ (resp. $\mathcal{U}_Y(Q)$) the set of Ulam-subgraphs of X (resp. of Y) isomorphic to Q.

We have to prove that, for any such Q

$$|\mathcal{U}_X(Q)| = |\mathcal{U}_Y(Q)|.$$

(indeed, this amounts to proving that X and Y have the same Ulam-deck, i.e. that (i) holds for some bijection $\pi : V(X) \to V(Y)$).

We shall split the proof into steps, according to the size of Q. We shall procede starting with the maximum size.

So, let l_X (resp. l_Y) be the largest size value of a subgraph of X (resp. of Y) of order $n-1$. By the assumption (ii) it is clear that $l_X = l_Y$: Indeed, if it were, say, $l_X < l_Y$, there would be in Y at least a subgraph U of order $n-1$ and size l_Y, hence $\binom{Y}{U} \geq 1$, whereas in X all subgraphs of order $n-1$ would have size $\leq l_X < l_Y$, hence $\binom{X}{U} = 0$, a contradiction. Thus we set $l := l_X = l_Y$.

Before proceeding, note the important fact that any subgraph of X (resp. of Y) of order $n-1$, and of arbitrary size s, is contained in *exactly one* Ulam-subgraph. In other words, there are no subgraphs of order $n-1$ in the intersection of two distinct Ulam-subgraphs (possibly of different sizes). Let Q be an arbitrary graph of order $n-1$. Denote by $\binom{G}{Q}^{[k]}$ the number of subgraphs of a graph G of order n isomorphic to Q and contained in some Ulam-subgraph of size k.

By the above consideration, it follows that

- if Q_l is a graph of (order $n-1$ and) size equal to l, then

$$\binom{X}{Q_l} = \binom{X}{Q_l}^{[l]} \quad \text{and} \quad \binom{Y}{Q_l} = \binom{Y}{Q_l}^{[l]}. \tag{1}$$

  Indeed, a subgraph of (order $n-1$ and) size l is necessarily contained in (in fact it is equal to) some Ulam-subgraph of size l.

- If Q_{l-1} is a graph of (order $n-1$ and) size equal to $l-1$, then

$$\begin{aligned} \binom{X}{Q_{l-1}} &= \binom{X}{Q_{l-1}}^{[l]} + \binom{X}{Q_{l-1}}^{[l-1]}, \\ \binom{Y}{Q_{l-1}} &= \binom{Y}{Q_{l-1}}^{[l]} + \binom{Y}{Q_{l-1}}^{[l-1]}. \end{aligned} \tag{2}$$

  Indeed, a subgraph of (order $n-1$ and) size $l-1$ is either contained in some Ulam-subgraph of size l, or in in some Ulam-subgraph of size $l-1$.

In general, if Q_s is a graph of (order $n-1$ and) size equal to s, we have

$$\binom{X}{Q_s} = \sum_{k=s}^{l} \binom{X}{Q_s}^{[k]} \quad \text{and} \quad \binom{Y}{Q_s} = \sum_{k=s}^{l} \binom{Y}{Q_s}^{[k]}. \tag{3}$$

We shall use the equalities (3) in succession, starting with $s = l$. We begin by considering (one-by-one) representatives of all graphs Q_l of (order $n-1$ and) size l. From (1) (that is (3) with $s = l$) and from assumption (ii), we obtain

$$\binom{X}{Q_l}^{[l]} = \binom{Y}{Q_l}^{[l]},$$

that is $|\mathcal{U}_X(Q_l)| = |\mathcal{U}_Y(Q_l)|$. This equality allows us to set up (at least) a one-to-one iso-correspondence μ_l (i.e. with corresponding objects isomorphic) between the Ulam-subgraphs of X of size l and those of Y. By "restriction", μ_l gives rise to one-to-one iso-correspondences $\mu_{l,r}$ between the set of subgraphs H_r of X of size r *contained in some* Ulam-subgraph of size l, and the analogous set of subgraphs K_r of Y (see Fig. 3, where $r = l - 1$, and the action of $\mu_{l,l-1}$ is drawn only partially).

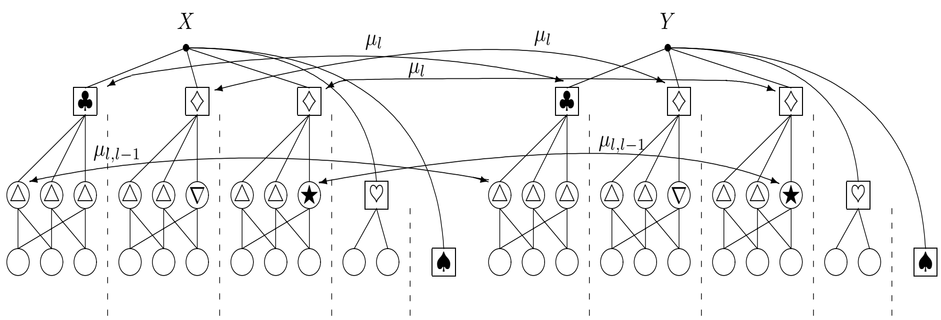

Figure 3: The one-to-one iso-correspondence $\mu_{l,l-1}$ induced by μ_l. The dashed vertical lines stress the fact that any subgraph of order $n - 1$ is contained in exactly one Ulam-subgraph (depicted in square frames).

Consequently we have, for any Q_r of (order $n - 1$ and) size $r < l$

$$\binom{X}{Q_r}^{[l]} = \binom{Y}{Q_r}^{[l]}. \tag{4}$$

Now, consider equality (2) (that is (3) with $s = l - 1$). Applying equality (4) with $r = l - 1$ and assumption (ii), we obtain

$$\binom{X}{Q_{l-1}}^{[l-1]} = \binom{Y}{Q_{l-1}}^{[l-1]},$$

that is $|\mathcal{U}_X(Q_{l-1})| = |\mathcal{U}_Y(Q_{l-1})|$. This equality allows us to set up (at least) a one-to-one iso-correspondence μ_{l-1} between the Ulam-subgraphs of X of size $l - 1$ and those of Y. By "restriction", μ_{l-1} gives rise to one-to-one iso-correspondences $\mu_{l-1,t}$ between the set of subgraphs H_t of X of size t *contained in some* Ulam-subgraph of size $l - 1$ and the analogous set of subgraphs K_t of Y.

Consequently we have, for any Q_t of (order $n-1$ and) size $t < l-1$

$$\binom{X}{Q_t}^{[l-1]} = \binom{Y}{Q_t}^{[l-1]}. \tag{5}$$

Now, consider equality (3) with $s = l-2$. Applying *both* equalities (4) with $r = l-2$ and (5) with $t = l-2$, together with assumption (ii), we obtain

$$\binom{X}{Q_{l-2}}^{[l-2]} = \binom{Y}{Q_{l-2}}^{[l-2]},$$

that is $|\mathcal{U}_X(Q_{l-2})| = |\mathcal{U}_Y(Q_{l-2})|$.

Repeating this argument for $l-3, \ldots, 1, 0$, we obtain the desired conclusion.

□

Remark 2 Because of the equivalence (i) $\Leftrightarrow$ (ii) (when $\mathcal{K} = \mathcal{G}$), Kelly–Ulam's conjecture can be rephrased by saying that two graphs of order n are isomorphic if and only if they contain the same number of subgraphs isomorphic to any graph Q of order less than n.

Although Conditions (ii) and (iii) of Theorem 1 are equivalent when considered for *all* graphs Q of order less than n, Conditions (ii) no longer implies Condition (iii) when Q is taken in a class $\mathcal{K}$ smaller than the class $\mathcal{G}$ of all graphs. Thus, for example, when the class $\mathcal{K}$ consists of the single graph K_2 (the connected graph on two vertices) Condition (ii) says that X and Y have the same number of edges, whereas Condition (iii) says that they have the same degree-sequence. As an example, one may take X to be a four cycle and Y a graph of order 4 having exactly one vertex of degree 1. The same X and Y also show that Condition (ii) does not imply Condition (iii) even if the class $\mathcal{K}$ consisted of all Q of order $n-2$.

However, since our proof of (ii) $\Rightarrow$ (i) only uses subgraphs of order $n-1$, we see that (ii) $\Rightarrow$ (iii) when $\mathcal{K}$ consists of all Q of order $n-1$.

5 $\mathcal{K}$-congruent pairs of graphs. Control-classes for a given class of graphs

Because of the equivalence of (i), (ii), and (iii) in Theorem 1 (when $\mathcal{K} = \mathcal{G}$), whenever the Kelly–Ulam's conjecture is proved for two graphs X, Y, such a result can be reformulated in two ways.

For example, Kelly's Theorem on trees ([7]) can be reformulated in the following ways (omitting the recognition part of his statement)

(O) *Let T_1, T_2 be two trees of order n. If for all graphs $Q \in \mathcal{G}$ of order less than n it holds $\binom{T_1}{Q} = \binom{T_2}{Q}$, then $T_1 \simeq T_2$.*

(P) *Let T_1, T_2 be two trees of order n. If there is a bijection $\pi : V(T_1) \to V(T_2)$ such that $\binom{T_1}{Q}_v = \binom{T_2}{Q}_{\pi(v)}$ for all $v \in V(T_1)$ and for all $Q \in \mathcal{G}$ of order less than n, then $T_1 \simeq T_2$.*

This leads to the following definitions.

Definition 2 Let $\mathcal{K}$ be a class of graphs, and X, Y graphs of order n. We say that X, Y are *$\mathcal{K}$-homogeneous* if $\binom{X}{Q} = \binom{Y}{Q}$ for all Q in $\mathcal{K}$ of order less than n. We say that X, Y are *$\mathcal{K}$-congruent* if there is a bijection $\pi : V(X) \to V(Y)$, called *$\mathcal{K}$-congruence*, such that $\binom{X}{Q}_v = \binom{Y}{Q}_{\pi(v)}$ for all $v \in V(X)$ and all Q in $\mathcal{K}$ of order less than n.

Definition 3 Let $\mathcal{K}$ be a class of graphs. The *$\mathcal{K}$-table* of a graph G is the array whose rows are labelled by the vertices of G, whose columns are labelled by representatives of the isomorphism classes of the graphs of $\mathcal{K}$ such that, for $v \in V(G)$, $Q \in \mathcal{K}$, the entry at position (v, Q) is the number of subgraphs of G containing v isomorphic to Q.

From this definition it follows that two graphs X and Y are $\mathcal{K}$-congruent if and only if their $\mathcal{K}$-tables are equal, up to reordering of the rows.

Definition 4 Let $\mathcal{A}$ be a class of graphs of order n. A class of graphs $\mathcal{K}$ is called an *overall control-class* for $\mathcal{A}$ if two graphs $G_1, G_2 \in \mathcal{A}$ are isomorphic whenever $\binom{G_1}{Q} = \binom{G_2}{Q}$, for all $Q \in \mathcal{K}$ of order less than n, i.e. whenever they are $\mathcal{K}$-homogeneous.

Similarly, $\mathcal{K}$ is called a *pointwise control-class* for $\mathcal{A}$ if two graphs $G_1, G_2 \in \mathcal{A}$ are isomorphic whenever there is a bijection $\pi : V(G_1) \to V(G_2)$ for which $\binom{G_1}{Q}_v = \binom{G_2}{Q}_{\pi(v)}$, for all $v \in V(G_1)$ and all $Q \in \mathcal{K}$ of order less than n, i.e. whenever they are $\mathcal{K}$-congruent.

Remark 3 Since

$$\binom{G}{Q} = \frac{1}{|Q|} \sum_{v \in V(G)} \binom{G}{Q}_v$$

then, clearly, if $\mathcal{K}$ is an overall control-class for $\mathcal{A}$, then $\mathcal{K}$ is also a control-class for $\mathcal{A}$. Moreover, if Kelly–Ulam's Conjecture is true, then the class $\mathcal{G}$ of all graphs is a control-class for any $\mathcal{A}$.

A generalization of the Reconstruction Problem which seems interesting to us is the following.

Problem 1 Find *minimal* control-classes for $\mathcal{A}$, when $\mathcal{A}$ is a class of reconstructible graphs, for instance the class of trees ([7]), cacti ([4], [10]), maximal planar graphs ([8]) and so on (minimal control-classes may not be unique).

In the special case when $\mathcal{A}$ is the class of all trees of a fixed order n, one may consider several interesting candidates for control-classes (either *overall* or *pointwise*)

- the class $\mathcal{P}$ of paths,
- the class $\mathcal{P}_\sigma$ of σ-paths (i.e. disjoint unions of paths),

- the class $\mathcal{C}$ of caterpillars,
- the class $\mathcal{C}_\sigma$ of σ-caterpillars,
- the class $\mathcal{O}$ of octopi (i.e. trees with at most one vertex of degree greater than 2).

Each class listed above has the feature that it contains the connected subgraphs of its elements. The classes $\mathcal{P}_\sigma$ and $\mathcal{C}_\sigma$ in fact contain all subgraphs of their elements.

It is not known if the classes $\mathcal{P}_\sigma$, $\mathcal{C}$, $\mathcal{C}_\sigma$, $\mathcal{O}$ listed above are pointwise control-classes for the trees. In [3] example-pairs are given that show that $\mathcal{P}$ is not a pointwise control-class for the trees of order n for many values of n. The minimal pair ($n = 20$) is shown in Figure 4. This pair also shows that $\mathcal{O}$ is not an overall control-class for the trees.

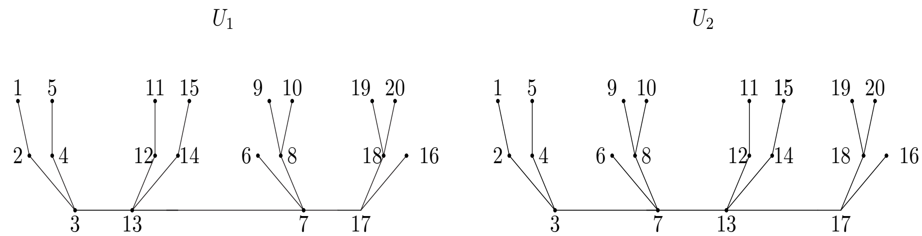

Figure 4: Minimal pair of non-isomorphic $\mathcal{P}$-congruent trees.

Remark 4 In view of the remark at the end of Section 4, if Kelly-Ulam's Conjecture is true, not only the class $\mathcal{G}$ of all graphs, but also the class $\mathcal{G}_{n-1}$ of all graphs of order exactly $n-1$ is an overall control-class for the class of all graphs of order n. However, in general, if $\mathcal{K}$ is a control-class for a class $\mathcal{A}$ of graphs of order n, it may not be true that also the class $\mathcal{K} \cap \mathcal{G}_{n-1}$ is a control-class for $\mathcal{A}$. In fact, $\mathcal{K} \cap \mathcal{G}_{n-1}$ may well be empty. For example, several trees of order n will contain no octopus or caterpillar of order $n-1$: Thus these trees could never be distinguished by the octopi or caterpillars of order $n-1$.

6 The Ulam-ladder

There are several ways of strengthening Kelly–Ulam's conjecture. The first and most natural is to ask whether fewer than n Ulam-subgraphs suffice to determine a graph (up to isomorphism). It has been proved that three suitably selected Ulam-subgraphs suffice "almost always" ([1]). For an arbitrary graph G of order n, Harary and Plantholt ([6]) have conjectured that $[\frac{n}{2}] + 2$ well-selected Ulam-subgraphs should suffice to determine G, and in fact 3 should suffice if n is prime.

To discuss another strengthening of Kelly–Ulam's conjecture we premise a definition.

Definition 5 The *Ulam-ladder* is the function $L : \mathbb{N} \to \mathbb{N}$ defined by setting $L(n)$ to be the smallest positive integers m such that all graphs of order n are determined by their induced subgraphs of order m.

There is some evidence to contend that

$$\lim_{n\to\infty} n - L(n) = \infty.$$

However, Nýdl has proved that for any fixed rational number $q < 1$, there is a positive integer n and a graph G of order n such that the knowledge of all induced subgraphs of G of order less than or equal to qn does not allow to determine G ([11]). In other words, if the Kelly–Ulam's conjecture is true, the graph of $L(n)$ lies below the straight line $y = x - 1$, but, by Nýdl's result, it does *not* lie below any straight line passing through the origin of slope $q < 1$. However, a shape for the graph of $L(n)$ like the one hinted at in Figure 5 would be compatible with Nýdl's result (the first eight values of $L(n)$ that we have drawn have been verified by computer).

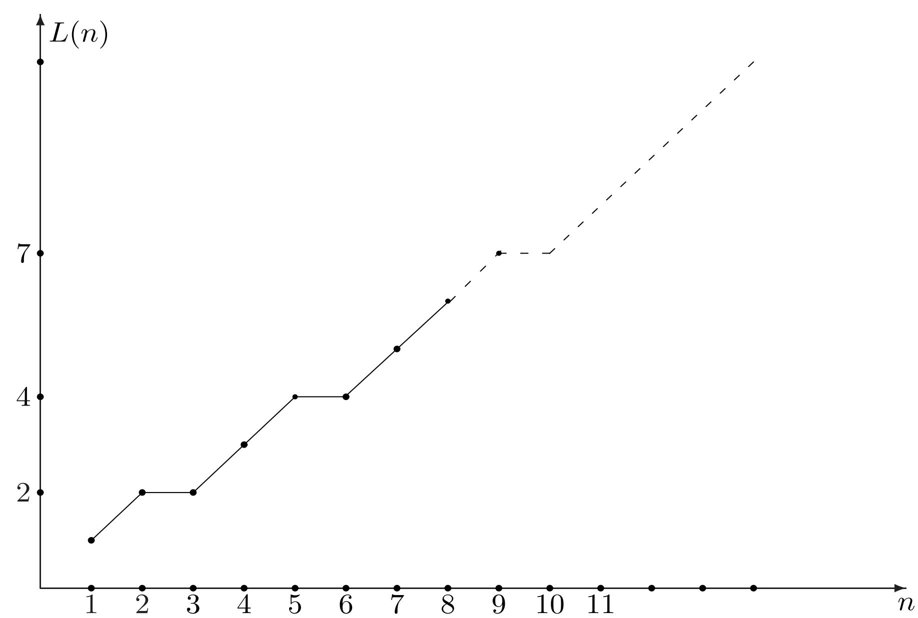

Figure 5: The Ulam-ladder.

We believe that the determination of the Ulam-ladder is one of the most charming problems in graph reconstruction.

References

[1] Bollobás, B.: *Almost every graph has reconstruction number three.* J. Graph Theory **14** (1990), 1–4

[2] Bondy, J. A.: A Graph Reconstructor's Manual. *Lecture Notes LMS, vol. 166, Cambridge Univ. Press*, 1991.

[3] Dulio, P., Pannone, V.: *Trees with the same path-table.* submitted.

[4] Geller, D., Manvel, B.: *Reconstruction of cacti.* Canad. J. Math. **21** (1969), 1354–1360.

[5] Greenwell, D. L., Hemminger, R. L.: *Reconstructing the n-connected components of a graph.* Aequationes Math. **9** (1973), 19–22.

[6] Harary, F., Plantholt, M.: *The Graph Reconstruction Number.* J. Graph Theory **9** (1985), 451–454.

[7] Kelly, P. J.: *A congruence Theorem for Trees.* Pacific J. Math. **7** (1957), 961–968.

[8] Lauri, J.: *The reconstruction of maximal planar graphs. II. Reconstruction.* J. Combin. Theory, Ser. B **30**, 2 (1981), 196–214.

[9] Lauri, J.: *Graph reconstruction-some techniques and new problems.* Ars Combinatoria, ser. B **24** (1987), 35–61.

[10] Monson, S. D.: *The reconstruction of cacti revisited.* Congr. Numer. **69** (1989), 157–166.

[11] Nýdl, V.: *Finite undirected graphs which are not reconstructible from their large cardinality subgraphs.* Discrete Math. **108** (1992), 373–377.

[12] Tutte, W. T.: *All the king's horses. A guide to reconstruction.* In: Graph Theory and Related Topics, *Acad. Press, New York*, 1979 (pp. 15–33).

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Continuous Dependence of Inverse Fundamental Matrices of Generalized Linear Ordinary Differential Equations on a Parameter

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Abstract

The problem of continuous dependence for inverses of fundamental matrices in the case when uniform convergence is violated is presented here.

Key words: Generalized linear ordinary differential equations, fundamental matrix, adjoint equation, continuous dependence on a parameter, emphatic convergence.

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1 Introduction

In this work we are dealing with the problem of continuous dependence for inverses of fundamental matrices. We make use of the results from [A] and from [T1, chapter 3].

In the second section a survey of known results concerning systems of generalized linear ordinary differential equations, fundamental matrix and adjoint equation is given. Main results of [A] and [T1, chapter 3] are presented here, too.

Our main result is formulated in Theorem 4. The case when uniform convergence is violated is presented here.

1.1 Preliminaries

The following notations and definitions will be used throughout this text: $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. $\mathbb{R}$ is the set of real numbers; $\mathbb{R}^{m\times n}$ is the space of real $m \times n$ matrices $B = (b_{ij})_{\substack{i=1,\dots,m\\ j=1,\dots,n}}$ with the norm

$$|B| = \max_{j=1,\dots,n} \sum_{i=1}^{m} |b_{ij}|\,;$$

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ stands for the set of real column n-vectors $b = (b_i)_{i=1}^n$.

For a matrix $B \in \mathbb{R}^{n\times n}$, $\det B$ denotes the determinant of B. If $\det B \neq 0$, then the matrix inverse to B is denoted by B^{-1}. B^T is the matrix transposed to B. The symbol I stands for the identity matrix and 0 for the zero matrix.

If $a, b \in \mathbb{R}$ are such that $-\infty < a < b < +\infty$, then $[a, b]$ stands for the closed interval $\{x \in \mathbb{R};\ a \le x \le b\}$, (a, b) is its interior and $(a, b]$, $[a, b)$ are the corresponding half-closed intervals.

The sets $D = \{t_0, t_1, t_2, \dots, t_m\}$ of points in the closed interval $[a, b]$ such that $a = t_0 < t_1 < t_2 < \dots < t_m = b$ are called divisions of $[a, b]$. The set of all divisions of the interval $[a, b]$ is denoted by $\mathcal{D}[a, b]$.

Let $B : [a, b] \to \mathbb{R}^{m\times n}$ be a matrix valued function. Its variation $\operatorname{var}_a^b B$ on the interval $[a, b]$ is defined by

$$\operatorname{var}_a^b B = \sup_{D\in\mathcal{D}[a,b]} \sum_{i=1}^{m} |B(t_i) - B(t_{i-1})|\,.$$

If $\operatorname{var}_a^b B < +\infty$, we say that the function B is of bounded variation on the interval $[a, b]$. $\mathbf{BV}^{m\times n}[a, b]$ denotes the set of all $m \times n$ matrix valued functions of bounded variation on $[a, b]$. We will write $\mathbf{BV}^n[a, b]$ instead of $\mathbf{BV}^{n\times 1}[a, b]$. For further details concerning the space $\mathbf{BV}^{m\times n}[a, b]$, see e.g. [T2].

We will write briefly $B(t+) = \lim_{\tau\to t+} B(\tau)$, $B(s-) = \lim_{\tau\to s-} B(\tau)$ and $\Delta^+ B(t) = B(t+) - B(t)$, $\Delta^- B(s) = B(s) - B(s-)$, $\Delta B(r) = B(r+) - B(r-)$ for $t \in [a, b)$, $s \in (a, b]$, $r \in (a, b)$.

If a sequence of $m \times n$ matrix valued functions $\{B_k(t)\}_{k=1}^\infty$ converges uniformly to a matrix valued function $B_0(t)$ on $[c, d] \subset [a, b]$, i.e.

$$\lim_{k\to\infty} \sup_{t\in[c,d]} |B_k(t) - B_0(t)| = 0,$$

we write

$$B_k \rightrightarrows B_0 \quad \text{on } [c, d].$$

We say that $\{B_k(t)\}_{k=1}^\infty$ converges locally uniformly to $B_0(t)$ on a set $M \subset [a, b]$, if $B_k \rightrightarrows B_0$ on each closed subinterval $J \subset M$.

We say that a proposition $P(n)$ holds for almost all (briefly a.a.) $n \in \mathbb{N}$ if it is true for all $n \in \mathbb{N} \setminus K$ where K is a finite set.

1.2 Kurzweil–Stieltjes integral

In this subsection we will recall the definition of the Kurzweil–Stieltjes integral (shortly KS-integral). We will work with the usual KS-integral which is equivalent to Perron–Stieltjes integral; cf. [STV, I.4.5], [T2, section 5].

Let $-\infty < a < b < +\infty$. For given $m \in \mathbb{N}$, a division $D = \{t_0, t_1, \dots, t_m\} \in \mathcal{D}[a,b]$ and $\xi = (\xi_1, \xi_2, \dots, \xi_m) \in \mathbb{R}^m$, the couple $P = (D, \xi)$ is called a partition of $[a,b]$ if

$$t_{j-1} \le \xi_j \le t_j \quad \text{for all } j = 1, 2, \dots, m.$$

The set of all partitions of the interval $[a,b]$ is denoted by $\mathcal{P}[a,b]$.

An arbitrary positive valued function $\delta : [a,b] \to (0, +\infty)$ is called a gauge on $[a,b]$. Given a gauge δ on $[a,b]$, the partition

$$P = (D, \xi) = \big(\{t_0, t_1, \dots, t_m\},\, (\xi_1, \xi_2, \dots, \xi_m)\big) \in \mathcal{P}[a,b]$$

is said to be δ-fine, if

$$[t_{j-1}, t_j] \subset \big(\xi_j - \delta(\xi_j), \xi_j + \delta(\xi_j)\big) \quad \text{for all } j = 1, 2, \dots, m.$$

The set of all δ-fine partitions of the interval $[a,b]$ is denoted by $\mathcal{A}(\delta; [a,b])$.

For functions $f, g : [a,b] \to \mathbb{R}$ and a partition $P \in \mathcal{P}[a,b]$,

$$P = \big(\{t_0, t_1, \dots, t_m\},\, (\xi_1, \xi_2, \dots, \xi_m)\big)$$

we define

$$S_P(f\,\Delta g) = \sum_{i=1}^{m} f(\xi_i)[g(t_i) - g(t_{i-1})].$$

We say, that $I \in \mathbb{R}$ is the KS-integral of f with respect to g from a to b if $\forall \varepsilon > 0\ \exists \delta : [a,b] \to (0,+\infty)\ \forall P \in \mathcal{A}(\delta;[a,b]) : |I - S_P(f\,\Delta g)| < \varepsilon$. In such a case we write $I = \int_a^b f\,\mathrm{d}g$ or $I = \int_a^b f(t)\,\mathrm{d}g(t)$.

It is known (cf. [T2, 5.20, 5.15]) that the KS-integral $\int_a^b f\,\mathrm{d}g$ exists, e.g. if $f \in \mathbf{BV}[a,b]$ and $g \in \mathbf{BV}[a,b]$. For the basic properties of the KS-integral, see [T2] and [STV].

If $F : [a,b] \to \mathbb{R}^{m\times n}$, $G : [a,b] \to \mathbb{R}^{n\times p}$ and $H : [a,b] \to \mathbb{R}^{p\times m}$ are matrix valued functions, then the symbols

$$\int_a^b F\,\mathrm{d}[G] \quad \text{and} \quad \int_a^b \mathrm{d}[H]\,F$$

stand for the matrices

$$\Big(\sum_{j=1}^{n} \int_a^b f_{ij}\,\mathrm{d}[g_{jk}]\Big)_{\substack{i=1,\dots,m\\ k=1,\dots,p}} \quad \text{and} \quad \Big(\sum_{i=1}^{m} \int_a^b f_{ki}\,\mathrm{d}[h_{ij}]\Big)_{\substack{k=1,\dots,p\\ j=1,\dots,n}},$$

whenever all the integrals appearing in the sums exist. Since the integral of a matrix valued function with respect to a matrix valued function is a matrix whose elements are sums of KS-integrals of real functions with respect to real functions, it is easy to reformulate all the statements from section 5 in [T2] for matrix valued functions (cf. [STV], I.4).

2 Generalized linear differential equations and the adjoint equation

Here we describe some fundamental properties of generalized linear differential equations, fundamental matrices and adjoint equations. More detailed information can be found in [STV]. We restrict ourselves to the interval $[0,1]$. The modification to the case of an arbitrary closed interval $[a,b] \subset \mathbb{R}$ in place of $[0,1]$ is evident.

2.1 Definition and basic properties

Assume that $A \in \mathbf{BV}^{n\times n}[0,1]$ and consider the equation

$$x(t) = x(s) + \int_s^t \mathrm{d}[A]\,x. \tag{2.1}$$

Let $[a,b] \subset [0,1]$. We say that a function $x : [a,b] \to \mathbb{R}^n$ is a solution of (2.1) on $[a,b]$ if there exists the KS-integral $\int_a^b \mathrm{d}[A]\,x \in \mathbb{R}^n$ and (2.1) holds for all $t,s \in [a,b]$.

Moreover, if $t_0 \in [a,b]$ and $\tilde{x} \in \mathbb{R}^n$ are given, we say that $x : [a,b] \to \mathbb{R}^n$ is a solution of the initial value problem (2.1), $x(t_0) = \tilde{x}$ on $[a,b]$ if it is a solution of (2.1) on $[a,b]$ and $x(t_0) = \tilde{x}$, i.e. if

$$x(t) = \tilde{x} + \int_{t_0}^t \mathrm{d}[A]\,x \tag{2.2}$$

for all $t \in [a,b]$.

Notice that, under the assumption $A \in \mathbf{BV}^{n\times n}[0,1]$, each solution of the equation (2.1) on $[0,1]$ is of bounded variation on $[0,1]$ (see [STV, III.1.3]).

Theorem 1 ([STV, III.1.4]) *Let $A \in \mathbf{BV}^{n\times n}[0,1]$. If $t_0 \in [0,1]$, then the initial value problem (2.2) possesses for any $\tilde{x} \in \mathbb{R}^n$ a unique solution $x(t)$ defined on $[0,1]$ if and only if $\det[\mathrm{I} - \Delta^- A(t)] \neq 0$ on $(t_0,1]$ and $\det[\mathrm{I} + \Delta^+ A(t)] \neq 0$ on $[0,t_0)$.*

2.2 Fundamental matrix

Lemma 1 ([STV, III.2.10, III.2.11]) *For a given $A \in \mathbf{BV}^{n\times n}[0,1]$ such that*

$$\det[\mathrm{I} - \Delta^- A(t)] \neq 0 \text{ on } (0,1] \text{ and } \det[\mathrm{I} + \Delta^+ A(t)] \neq 0 \text{ on } [0,1) \tag{2.3}$$

there exists a unique $U : [0,1] \times [0,1] \to \mathbb{R}^{n\times n}$ such that

$$U(t,s) = \mathrm{I} + \int_s^t \mathrm{d}[A(r)]\,U(r,s)$$

for all $t,s \in [0,1]$.

Moreover, there exists a unique matrix valued function $X : [0,1] \to \mathbb{R}^{n\times n}$ *such that* $\det X(t) \neq 0$ *for* $t \in [0,1]$,

$$U(t,s) = X(t)\, X^{-1}(s) \quad \textit{for all } \ s,t \in [0,1] \tag{2.4}$$

and

$$X(t) = \mathrm{I} + \int_0^t \mathrm{d}[A]\, X,\ t \in [0,1]\,. \tag{2.5}$$

Furthermore, the inverse matrix $X^{-1}(t)$ *is of bounded variation on* $[0,1]$ *and it satisfies the relation*

$$X^{-1}(t) = X^{-1}(s) - X^{-1}(t)\, A(t) + X^{-1}(s)\, A(s) + \int_s^t \mathrm{d}[X^{-1}]\, A \tag{2.6}$$

for $t,s \in [0,1]$.

For a given $t_0 \in [0,1]$, the unique solution $x(t)$ of (2.2) on $[t_0,1]$ (see Theorem 1) is given by

$$x(t) = X(t)\, X^{-1}(t_0)\, \tilde{x}.$$

Definition 1 The matrix $X : [0,1] \to \mathbb{R}^{n\times n}$ given by Lemma 1 is called the fundamental matrix of the homogenous generalized linear differential equation (2.1) or briefly the fundamental matrix corresponding to the given matrix function A.

2.3 Adjoint equation

The equation (2.6), which is satisfied by the matrix function X^{-1}, is not a generalized linear differential equation of the type (2.1). This leads us to the consideration of adjoint equations, i.e. the equations of the form

$$y^T(t) = y^T(s) - y^T(t)\, A(t) + y^T(s)\, A(s) + \int_s^t \mathrm{d}[y^T]\, A\,. \tag{2.7}$$

Theorem 2 ([ST, 2.7]) *Let* $A \in \mathbf{BV}^{n\times n}[0,1]$ *satisfy* (2.3). *Then the initial value problem* (2.7), $y^T(1) = \tilde{y}^T$ *has for every* $\tilde{y} \in \mathbb{R}^n$ *a unique solution* $y : [0,1] \to \mathbb{R}^n$ *on* $[0,1]$. *This solution is of bounded variation on* $[0,1]$ *and is given on* $[0,1]$ *by*

$$y^T(s) = \tilde{y}^T X(1) X^{-1}(s). \tag{2.8}$$

Moreover, every solution $y^T(t)$ *of the equation (2.7) on* $[0,1]$ *possesses the onesided limits* $y^T(t+)$, $y^T(t-)$ *where the relations*

$$\begin{aligned} y^T(t+) &= y^T(t) - y^T(t+)\,\Delta^+ A(t) \quad \textit{for all } t \in [0,1)\,,\\ y^T(t-) &= y^T(t) + y^T(t-)\,\Delta^- A(t) \quad \textit{for all }, t \in (0,1] \end{aligned} \tag{2.9}$$

hold.

2.4 Convergence results for generalized linear ordinary differential equations

In [T1, Theorem 3.3.2] the continuous dependence of the fundamental matrix X of (2.1) on a parameter was described. Let us recall this result. To this aim we need the following notations.

Notation 1 Let a sequence $\{A_k\}_{k=1}^{\infty} \subset \mathbf{BV}^{n\times n}[0,1]$ and $A_0 \in \mathbf{BV}^{n\times n}[0,1]$. For a $k \in \mathbb{N}$ and an arbitrary closed interval $J = [\alpha, \beta] \subset [0,1]$, define

$$A_k^J(t) = A_k(t) - A_k(\alpha) \quad \text{for } k \in \mathbb{N}_0,\ t \in J.$$

Theorem 3 ([T1, Theorem 3.3.2]) *Let* $A_k \in \mathbf{BV}^{n\times n}[0,1]$ *for* $k \in \mathbb{N}_0$ *and* $\det[\mathrm{I} - \Delta^- A_0(t)] \neq 0$ *on* $(0,1]$. *Furthermore, assume that there is a finite set* $D \subset [0,1]$ *such that:*

$$A_k^J(s) \rightrightarrows A_0^J(s) \text{ on } J \text{ for any closed interval } J \subset [0,1]\setminus D, \tag{2.10}$$

$$\sup_{k\in\mathbb{N}} \operatorname{var} A_k < +\infty \text{ and } \det[\mathrm{I} - \Delta^- A_k(t)] \neq 0 \text{ for all } t \in D \text{ and for a.a. } k \in \mathbb{N}, \tag{2.11}$$

$$\left.\begin{array}{l}
\text{if } \tau \in D, \text{ then } \forall \xi \in \mathbb{R}^n \text{ and } \forall \varepsilon > 0\ \exists \delta > 0 \text{ such that} \\
\forall \delta' \in (0,\delta)\ \exists k_0 \in \mathbb{N} \text{such that the relations} \\
\qquad |u_k(\tau) - u_k(\tau - \delta') - \Delta^- A_0(\tau)[\mathrm{I} - \Delta^- A_0(\tau)]^{-1}\xi| < \varepsilon, \\
\qquad |v_k(\tau + \delta') - v_k(\tau) - \Delta^+ A_0(\tau)\xi| < \varepsilon \\
\text{are satisfied } \forall k \geq k_0 \text{ and } \forall u_k, v_k \text{ such that} \\
\qquad |\xi - u_k(\tau - \delta')| \leq \delta,\ |\xi - v_k(\tau)| \leq \delta \text{ and} \\
\qquad u_k(t) = u_k(\tau - \delta') + \displaystyle\int_{\tau-\delta'}^t \mathrm{d}[A_k]\, u_k(s) \quad \text{on } [\tau - \delta', \tau], \\
\qquad v_k(t) = v_k(\tau) + \displaystyle\int_{\tau}^t \mathrm{d}[A_k]\, v_k(s) \quad \text{on } [\tau, \tau + \delta'].
\end{array}\right\} \tag{2.12}$$

Then for a.a. $k \in \mathbb{N}$ *the fundamental matrix* X_k *corresponding to* A_k *is defined on* $[0,1]$ *and*

$$\lim_{k\to\infty} X_k(t) = X_0(t) \quad \text{on } [0,1]. \tag{2.13}$$

A similar assertion concerning inverses of fundamental matrices will be proved in Theorem 4.

Remark 1 Theorem 3 is a slightly modified version of [T1, Theorem 3.3.2]. Notation is simplified and, in particular, from the proof given in [T1, Theorem 3.3.2] it follows that the assumption $\det[\mathrm{I} - \Delta^- A_k(t)] \neq 0$ on $(0,1]$ for all $k \in \mathbb{N}$ used in [T1] is not necessary and it can be replaced by a weaker one, i.e. $\det[\mathrm{I} - \Delta^- A_k(t)] \neq 0$ for all $t \in D$, for a.a. $k \in \mathbb{N}$.

Conditions (2.10)–(2.12) characterize the concept of emphatic convergence introduced by J. Kurzweil (cf. [K2, Definition 4.1]). For more details see [T1, Definition 3.2.8] or [S].

In the proof of Theorem 4 the following two lemmas are needed. The former one is from [A, Lemma 2]. The latter one is based on [T1, Theorem 3.2.5] and on [A, Lemma 2].

Lemma 2 ([A, Lemma 2]) *Let* $-\infty < a < b < +\infty$*,* $A_k \in \mathbf{BV}^{n\times n}[a,b]$ *for* $k \in \mathbb{N}_0$ *and let* $\det[\mathrm{I}+\Delta^+ A_0(t)] \neq 0$ *on* $[a,b)$ *and* $\det[\mathrm{I}-\Delta^- A_0(t)] \neq 0$ *on* $(a,b]$*. If* $X_k \rightrightarrows X_0$ *on* $[a,b]$*, then* $X_k^{-1} \rightrightarrows X_0^{-1}$ *on* $[a,b]$*.*

Lemma 3 *Let* $-\infty < a < b < +\infty$*,* $A_k \in \mathbf{BV}^{n\times n}[a,b]$ *for* $k \in \mathbb{N}_0$ *and* $\det[\mathrm{I}+\Delta^+ A_0(t)] \neq 0$ *on* $[a,b)$ *and* $\det[\mathrm{I}-\Delta^- A_0(t)] \neq 0$ *on* $(a,b]$*. Assume that the sequence* $\{A_k\}_{k=1}^{\infty}$ *satisfies the following two conditions*

(i) $\sup\limits_{k\in\mathbb{N}} \operatorname{var}_a^b A_k < +\infty$*,*

(ii) $[A_k(t) - A_k(a)] \rightrightarrows [A_0(t) - A_0(a)]$ *on* $[a,b]$*.*

Then for $k = 0$ *and for a.a.* $k \in \mathbb{N}$ *there exists the fundamental matrix* X_k *corresponding to* A_k *on* $[a,b]$ *and* $X_k^{-1} \rightrightarrows X_0^{-1}$ *on* $[a,b]$*.*

3 Main result

Theorem 3 deals with a sequence of fundamental matrices. According to definition, each fundamental matrix corresponding to a given matrix function A fulfills for all $s,t \in [0,1]$ the equation

$$X(t) = X(s) + \int_s^t \mathrm{d}[A]\, X.$$

This fact is essentially used in the proof of Theorem 4. Furthermore, we take into account that the inverse of fundamental matrix $X^{-1}(t)$ satisfies relation

$$X^{-1}(t) = X^{-1}(0) - X^{-1}(t)\, A(t) + X^{-1}(0)\, A(0) + \int_0^t \mathrm{d}[X^{-1}]\, A, \qquad (3.14)$$

which is adjoint to (2.5), see (2.6) and (2.7).

We want to prove assertion analogous to Theorem 3 for inverses of fundamental matrices. To this aim it is necessary to suppose also the regularity of $[\mathrm{I}+\Delta^+ A_0(t)]$ for each $t \in [0,1)$ and the condition (3.15) which is a modification of (2.12) for relation (3.14). This is our main result.

Theorem 4 *Let the assumptions of Theorem* 3 *are satisfied. Furthermore assume that* $\det[\mathrm{I} + \Delta^+ A_0(t)] \neq 0$ *on* $[0,1)$ *and the following conditions hold:*

$$\left.\begin{aligned}
&\textit{if } \tau \in D, \textit{ then } \forall \eta \in \mathbb{R}^n \textit{ and } \forall \varepsilon > 0\ \exists \delta > 0\\
&\textit{such that } \forall \delta' \in (0,\delta)\ \exists k_0 \in \mathbb{N} \textit{such that the relations}\\
&\quad |w_k^T(\tau) - w_k^T(\tau - \delta') + \eta^T\, \Delta^- A_0(\tau)| < \varepsilon\,,\\
&\quad |z_k^T(\tau + \delta') - z_k^T(\tau) + \eta^T\, [\mathrm{I} + \Delta^+ A_0(\tau)]^{-1}\, \Delta^+ A_0(\tau)| < \varepsilon\\
&\textit{are satisfied } \forall k \geq k_0 \textit{ and } \forall w_k,\ z_k \in \mathbb{R}^n \textit{ fulfilling (3.16), (3.17)}\\
&\textit{and such that}\\
&\quad |\eta^T - w_k^T(\tau - \delta')| \leq \delta,\ |\eta^T - z_k^T(\tau)| \leq \delta,
\end{aligned}\right\} \tag{3.15}$$

where

$$\begin{aligned} w_k^T(t) \;=\; & w_k^T(\tau - \delta') - w_k^T(t) A_k(t) + w_k^T(\tau - \delta') A_k(\tau - \delta')\\ & + \int_{\tau - \delta'}^t \mathrm{d}[w_k^T]\, A_k \;\; \textit{on } [\tau - \delta', \tau], \end{aligned} \tag{3.16}$$

$$z_k^T(t) = z_k^T(\tau) - z_k^T(t)\, A_k(t) + z_k^T(\tau) A_k(\tau) + \int_\tau^t \mathrm{d}[z_k^T]\, A_k \;\; \textit{on } [\tau, \tau + \delta']. \tag{3.17}$$

Then for a.a. $k \in \mathbb{N}$ *the fundamental matrices* X_k *corresponding to* A_k *and their inverses* X_k^{-1} *are defined on* $[0,1]$,

$$\lim_{k \to \infty} X_k(t) = X_0(t) \quad \textit{on } [0,1] \tag{3.18}$$

and

$$\lim_{k \to \infty} X_k^{-1}(t) = X_0^{-1}(t) \quad \textit{on } [0,1]. \tag{3.19}$$

Moreover, (3.19) *holds locally uniformly on* $[0,1] \setminus D$.

Proof First notice that Lemma 3 implies that (3.19) holds locally uniformly on $[0,1] \setminus D$ and (3.18) immediately follows from Theorem 3.

Assume that $D = \{\tau\}$, where $\tau \in (0,1)$; i.e. D consists of one point $\tau \in (0,1)$ only and $m = 1$.

Recall that the existence of the fundamental matrices X_k for a.a. $k \in \mathbb{N}$ and (3.18) immediately follows from Theorem 3. Since each fundamental matrix is regular, we get the existence of X_k^{-1} for a.a. $k \in \mathbb{N}$. For $\tilde{y} \in \mathbb{R}^n$ and for a.a. $k \in \mathbb{N}_0$, denote by y_k^T the solution of the equation

$$y_k^T(t) = \tilde{y}^T - y_k^T(t)\, A_k(t) + \tilde{y}^T\, A_k(0) + \int_0^t \mathrm{d}[y_k^T]\, A_k \quad \text{on } [0,1]. \tag{3.20}$$

The rest of the proof splits into three steps. First, we prove that (3.19) is true for $t \in [0,\tau)$, then for $t = \tau$ and finally for $t \in (\tau, 1]$.

• Step 1. Let $\alpha \in (0,\tau)$ be given. Then by Lemma 3 the relation (3.19) holds uniformly on $[0,\alpha]$. Therefore (3.19) is true for any $t \in [0,\tau)$.

• Step 2. Now we will prove, that (3.19) is true also for $t = \tau$. For each $\delta^{'} \in (0,\tau)$ and $k \in \mathbb{N}$ we get using (2.9) the estimate

$$\begin{aligned} |y_0^T(\tau) - y_k^T(\tau)| &\le |y_0^T(\tau) + y_0^T(\tau-)\,\Delta^- A_0(\tau) - y_0^T(\tau - \delta^{'})| \\ + |y_0^T(\tau-\delta^{'}) - y_k^T(\tau-\delta^{'})| &+ |y_k^T(\tau-\delta^{'}) - y_0^T(\tau-)\,\Delta^- A_0(\tau) - y_k^T(\tau)| \\ &= |y_0^T(\tau-) - y_0^T(\tau-\delta^{'})| + |y_0^T(\tau-\delta^{'}) - y_k^T(\tau-\delta^{'})| \\ &\quad + |y_k^T(\tau) - y_k^T(\tau-\delta^{'}) + y_0^T(\tau-)\,\Delta^- A_0(\tau)|\,. \end{aligned}$$

Let $\varepsilon > 0$ be given. According to (3.15) we can choose $\delta \in (0,\varepsilon)$ in such a way that for all $\delta^{'} \in (0,\delta)$ there exists $k_1 \in \mathbb{N}$ with the property

$$|w_k^T(\tau) - w_k^T(\tau-\delta^{'}) + y_0^T(\tau-)\,\Delta^- A_0(\tau)| < \varepsilon \tag{3.21}$$

holds for any $k \ge k_1$ and for each solution $w_k^T(t)$ of (3.16) fulfilling

$$|y_0^T(\tau-) - w_k^T(\tau-\delta^{'})| \le \delta.$$

Set $w_k^T(t) = y_k^T(t)$ on $[\tau-\delta^{'},\tau]$. Choose $\delta^{'} \in (0,\delta)$ so that

$$|y_0^T(\tau-) - y_0^T(\tau-\delta^{'})| < \frac{\delta}{2}\,.$$

Considering that $y_k^T(t) \to y_0^T(t)$ on $[0,\tau)$ as $k \to \infty$ we get the existence of a $k_0 \in \mathbb{N}$, $k_0 \ge k_1$ such that $|y_0^T(\tau-\delta^{'}) - y_k^T(\tau-\delta^{'})| < \frac{\delta}{2}$ for all $k \ge k_0$. Therefore the estimate

$$|y_0^T(\tau-) - y_k^T(\tau-\delta^{'})| \le |y_0^T(\tau-) - y_0^T(\tau-\delta^{'})| + |y_0^T(\tau-\delta^{'}) - y_k^T(\tau-\delta^{'})| < \delta$$

is true for $k \ge k_0$. By (3.21) we have

$$|y_k^T(\tau) - y_k^T(\tau-\delta^{'}) + y_0^T(\tau-)\,\Delta^- A_0(\tau)| < \varepsilon.$$

To summarize, we have

$$|y_0^T(\tau) - y_k^T(\tau)| < \frac{\delta}{2} + \frac{\delta}{2} + \varepsilon < 2\,\varepsilon \quad \text{for all } k \ge k_0\,,$$

i.e. $y_k^T(\tau) \to y_0^T(\tau)$ for $k \to \infty$.

• Step 3. Proof of the convergence on $(\tau,1]$ consists of two parts. First, we show that there is a $\delta > 0$ such that $y_k^T(t) \to y_0^T(t)$ on $(\tau,\tau+\delta)$ as $k \to \infty$. Then we extend this result to the whole interval $(\tau,1]$. Let $\varepsilon > 0$ be given and let $\delta_0 \in (0,\varepsilon)$ be such that

$$|y_0^T(s) - y_0^T(\tau+)| < \varepsilon \quad \text{for all } s \in (\tau,\tau+\delta_0).$$

By the assumption (3.15), there exists $\delta \in (0,\delta_0)$ such that for all $\delta^{'} \in (0,\delta)$ there exists $k_1 = k_1(\delta^{'}) \in \mathbb{N}$ and such that

$$|z_k^T(\tau+\delta^{'}) - z_k^T(\tau) + y_0^T(\tau)\,[\mathrm{I} + \Delta^+ A_0(\tau)]^{-1}\,\Delta^+ A_0(\tau)| < \varepsilon \tag{3.22}$$

is true for each solution $z_k^T(t)$ of (3.17) with the property $|y_0^T(\tau) - z_k^T(\tau)| \le \delta$. Now the distance between $y_0^T(\tau+\delta^{'})$ and $y_k^T(\tau+\delta^{'})$ can be estimated. In view of (2.9) we get

$$|y_0^T(\tau+\delta^{'}) - y_k^T(\tau+\delta^{'})| \le |y_0^T(\tau+\delta^{'}) - y_0^T(\tau) + y_0^T(\tau+)\,\Delta^+ A_0(\tau)|$$
$$+ |y_0^T(\tau) - y_k^T(\tau)| + |y_k^T(\tau) - y_0^T(\tau+)\,\Delta^+ A_0(\tau) - y_k^T(\tau+\delta^{'})|$$
$$= |y_0^T(\tau+\delta^{'}) - y_0^T(\tau+)| + |y_0^T(\tau) - y_k^T(\tau)| + |y_k^T(\tau) - y_0^T(\tau+)\,\Delta^+ A_0(\tau) - y_k^T(\tau+\delta^{'})|.$$

Considering that $y_k^T(\tau) \to y_0^T(\tau)$ for $k \to \infty$, we get the existence of $k_0 \in \mathbb{N}$, $k_0 \ge k_1$ such that $|y_0^T(\tau) - y_k^T(\tau)| < \delta$ for all $k \ge k_0$. Since $\tau + \delta^{'} \in (\tau, \tau+\delta_0)$, we have $|y_0^T(\tau+\delta^{'}) - y_0^T(\tau+)| < \varepsilon$. Setting $z_k^T(t) = y_k^T(t)$ on $[\tau, \tau+\delta^{'}]$, we get by (3.22) the relation

$$|y_k^T(\tau) - y_0^T(\tau+)\,\Delta^+ A_0(\tau) - y_k^T(\tau+\delta^{'})| < \varepsilon \quad \text{for all } k \ge k_0\,.$$

To summarize, for any $k \ge k_0$ the estimate

$$|y_0^T(\tau+\delta^{'}) - y_k^T(\tau+\delta^{'})| \le \varepsilon + \delta + \varepsilon < 3\,\varepsilon$$

is valid, as well. Therefore $y_k^T(t) \to y_0^T(t)$ on $(\tau, \tau+\delta)$ as $k \to \infty$. Now, choose an arbitrary σ in $(\tau, \tau+\delta)$. Making use of Lemma 3 with $[a,b] = [\sigma, 1]$ the proof of this step can be completed.

Having solution $y_k^T(t)$ to (3.20) for each $\tilde{y} \in \mathbb{R}^n$, we can determine the matrix function $X_k^{-1}(t)$ from $y_k^T(t)$ using (2.8). Indeed, since $X_k(1)$ is regular, we can choose $\tilde{y}^T$ in such a way that $y_k^T(t)$ is i-th row of $X_k^{-1}(t)$. This consideration completes the proof of the validity of (3.19) for any $t \in [0,1]$.

The extension to the case $m > 1$ is obvious. □

References

[A] Ashordia, M.: *On the correctness of linear boundary value problems for systems of generalized ordinary differential equations.* Proceedings of the Georgian Academy of Sciences, Math. **1**, 4 (1993), 385–394.

[K1] Kurzweil, J.: *Generalized ordinary differential equations and continuous dependence on a parameter.* Czech. Math. J. **7** (82) (1957), 418–449.

[K2] Kurzweil, J.: *Generalized ordinary differential equations.* Czech. Math. J. **24** (83) (1958), 360–387.

[S] Schwabik, Š.: Generalized Ordinary Differential Equations. *World Scientific, Singapore*, 1992.

[STV] Schwabik, Š., Tvrdý, M., Vejvoda, O.: Differential and Integral Equations: Boundary Value Problems and Adjoints. *Academia and D. Reidel, Praha and Dordrecht*, 1979.

[ST] Schwabik, Š., Tvrdý, M.: *Boundary value problems for generalized linear differential equations.* Czech. Math. J. **29** (104) (1979), 451–477.

[T1] Tvrdý, M.: *Differential and integral equations in the space of regulated functions.* In: Memoirs on Differential Equations and Mathematical Physics, vol. 25, 2002, 1–104.

[T2] Tvrdý, M.: *Stieltjesův integrál.* Učební text [online, cited 22. 6. 2005], `http://www.math.cas.cz/~tvrdy/teaching.html`.

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Tests in Weakly Nonlinear Regression Model *

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Abstract

In weakly nonlinear regression model a weakly nonlinear hypothesis can be tested by linear methods if an information on actual values of model parameters is at our disposal and some condition is satisfied. In other words we must know that unknown parameters are with sufficiently high probability in so called linearization region. The aim of the paper is to determine this region.

Key words: Regression model, nonlinear hypothesis, linearization.

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0 Introduction

A nonlinear hypothesis on model parameters in nonlinear regression model can be tested by linear methods if some conditions are satisfied. This condition is given in the form of the inclusion $\mathcal{E} \subset \mathcal{L}_T$ which must occur with sufficiently high probability. Here $\mathcal{E}$ is the $(1-\alpha)$-confidence region of the model parameters (for sufficiently small α) and $\mathcal{L}_T$ is a special set in parameter space. The aim of the paper is to determine the set $\mathcal{L}_T$ (linearization region).

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1 Notation

Let $\mathbf{Y} \sim N_n[\mathbf{f}(\boldsymbol{\beta}), \sigma^2\mathbf{V}]$ be the regression model under consideration. Here $\mathbf{Y}$ is the n-dimensional normally distributed observation vector, $\mathbf{f}(\boldsymbol{\beta})$ is the mean value of the vector $\mathbf{Y}$, $\boldsymbol{\beta}$ is an unknown k-dimensional parameter, $\sigma^2\mathbf{V}$ is the covariance matrix of the vector $\mathbf{Y}$, σ^2 is known/unknown parameter and $\mathbf{V}$ is a given $n \times n$ positive definite matrix. The null hypothesis H_0 is given in the form $\mathbf{t}(\boldsymbol{\beta}) = \mathbf{0}$ and the alternative is $H_a : \mathbf{t}(\boldsymbol{\beta}) \neq \mathbf{0}$.

The functions $\mathbf{f}(\cdot)$ and $\mathbf{t}(\cdot)$ can be given in the form

$$\mathbf{f}(\boldsymbol{\beta}) = \mathbf{f}(\boldsymbol{\beta}^{(0)}) + \mathbf{F}\delta\boldsymbol{\beta} + \frac{1}{2}\boldsymbol{\kappa}(\delta\boldsymbol{\beta}), \quad \mathbf{t}(\boldsymbol{\beta}) = \mathbf{t}(\boldsymbol{\beta}^{(0)}) + \mathbf{T}\delta\boldsymbol{\beta} + \frac{1}{2}\boldsymbol{\tau}(\delta\boldsymbol{\beta}),$$

where $\delta\boldsymbol{\beta} = \boldsymbol{\beta} - \boldsymbol{\beta}^{(0)}$, $\boldsymbol{\beta}^{(0)}$ is an approximate value of the parameter $\boldsymbol{\beta}$,

$$\begin{aligned}
\mathbf{F} &= \left.\frac{\partial \mathbf{f}(\mathbf{u})}{\partial \mathbf{u}}\right|_{u=\beta^{(0)}}, \qquad \mathbf{T} = \left.\frac{\partial \mathbf{t}(\mathbf{u})}{\partial \mathbf{u}}\right|_{u=\beta^{(0)}}, \\
\boldsymbol{\kappa}(\delta\boldsymbol{\beta}) &= [\kappa_1(\delta\boldsymbol{\beta}), \ldots, \kappa_n(\delta\boldsymbol{\beta})]', \\
\kappa_i(\delta\boldsymbol{\beta}) &= (\delta\boldsymbol{\beta})' \left.\frac{\partial^2 f_i(\mathbf{u})}{\partial \mathbf{u} \partial \mathbf{u}'}\right|_{u=\beta^{(0)}} \delta\boldsymbol{\beta}, \quad i = 1, \ldots, n, \\
\boldsymbol{\tau}(\delta\boldsymbol{\beta}) &= [\tau_1(\delta\boldsymbol{\beta}), \ldots, \tau_q(\delta\boldsymbol{\beta})]', \\
\tau_i(\delta\boldsymbol{\beta}) &= (\delta\boldsymbol{\beta})' \left.\frac{\partial^2 t_i(\mathbf{u})}{\partial \mathbf{u} \partial \mathbf{u}'}\right|_{u=\beta^{(0)}} \delta\boldsymbol{\beta}, \quad i = 1, \ldots, q.
\end{aligned}$$

Let the rank of the $n \times k$ matrix $\mathbf{F}$ be $r(\mathbf{F}) = k < n$ and the rank of the $q \times k$ matrix $\mathbf{T}$ be $r(\mathbf{T}) = q < k$.

2 Determination of the region $\mathcal{L}_T$

The linearized form of the model and the hypothesis is

$$\mathbf{Y} - \mathbf{f}_0 \sim N_n(\mathbf{F}\delta\boldsymbol{\beta}, \sigma^2\mathbf{V}), \quad \mathbf{T}\delta\boldsymbol{\beta} = \mathbf{0}. \tag{1}$$

(The vector $\boldsymbol{\beta}^{(0)}$ should be chosen such that $\mathbf{t}(\boldsymbol{\beta}^{(0)}) = \mathbf{0}$.)

The quadratized form of the model and the hypothesis is

$$\mathbf{Y} - \mathbf{f}_0 \sim N_n\left(\mathbf{F}\delta\boldsymbol{\beta} + \frac{1}{2}\boldsymbol{\kappa}(\delta\boldsymbol{\beta}), \sigma^2\mathbf{V}\right), \quad \mathbf{T}\delta\boldsymbol{\beta} + \frac{1}{2}\boldsymbol{\tau}(\delta\boldsymbol{\beta}) = \mathbf{0}. \tag{2}$$

Lemma 2.1 *If the model (1) is valid, the test of the hypothesis is*

$$(\widehat{\delta\boldsymbol{\beta}})'\mathbf{T}'[\mathbf{T}(\mathbf{F}'\mathbf{V}^{-1}\mathbf{F})^{-1}\mathbf{T}]^{-1}\mathbf{T}\widehat{\delta\boldsymbol{\beta}} \sim \begin{cases} \sigma^2\chi_q^2(0) & \text{if } H_0 \text{ is true,} \\ \sigma^2\chi_q^2(\delta) & \text{if } H_0 \text{ is not true.} \end{cases}$$

Here $\widehat{\delta\boldsymbol{\beta}} = (\mathbf{F}'\mathbf{V}^{-1}\mathbf{F})^{-1}\mathbf{F}'\mathbf{V}^{-1}(\mathbf{Y} - \mathbf{f}_0)$ *and the parameter of noncentrality* δ *is*

$$\delta = [E(\widehat{\delta\boldsymbol{\beta}})]'\mathbf{T}'[\mathbf{T}(\mathbf{F}'\mathbf{V}^{-1}\mathbf{T})^{-1}\mathbf{T}']^{-1}\mathbf{T}E(\widehat{\delta\boldsymbol{\beta}})/\sigma^2.$$

Proof Cf. [4], chpt. 4. □

Lemma 2.2 *If the model (2) is valid, then under the null hypothesis* H_0 : $\mathbf{T}\delta\boldsymbol{\beta} + \frac{1}{2}\boldsymbol{\tau}(\delta\boldsymbol{\beta}) = \mathbf{0}$, *it is valid*

$$(\widehat{\delta\boldsymbol{\beta}})'\mathbf{T}'[\mathbf{T}(\mathbf{F}'\mathbf{V}^{-1}\mathbf{F})^{-1}\mathbf{T}]^{-1}\mathbf{T}\widehat{\delta\boldsymbol{\beta}} \sim \sigma^2\chi^2_q(\Delta).$$

Here

$$\Delta = \frac{1}{\sigma^2}\Big[-\frac{1}{2}\boldsymbol{\tau}(\mathbf{K}_T\delta\mathbf{u}) + \mathbf{T}(\mathbf{F}'\mathbf{V}^{-1}\mathbf{F})^{-1}\mathbf{F}'\mathbf{V}^{-1}\frac{1}{2}\boldsymbol{\kappa}(\mathbf{K}_T\delta\mathbf{u})\Big]'$$
$$\times\Big[\mathbf{T}(\mathbf{F}'\mathbf{V}^{-1}\mathbf{F})^{-1}\mathbf{T}'\Big]^{-1}\Big[-\frac{1}{2}\boldsymbol{\tau}(\mathbf{K}_T\delta\mathbf{u}) + \mathbf{T}(\mathbf{F}'\mathbf{V}^{-1}\mathbf{F})^{-1}\mathbf{F}'\mathbf{V}^{-1}\frac{1}{2}\boldsymbol{\kappa}(\mathbf{K}_T\delta\mathbf{u})\Big]$$

and $\delta\boldsymbol{\beta} = \mathbf{K}_T\delta\mathbf{u} +$ *terms of higher orders. The* $k\times(k-q)$ *matrix* $\mathbf{K}_T$ *is of the rank* $r(\mathbf{K}_T) = k-q$ *and it satisfies the equality* $\mathbf{T}\mathbf{K}_T = \mathbf{0}$.

Proof In model (2) the mean value of the estimator

$$\widehat{\delta\boldsymbol{\beta}} = (\mathbf{F}'\mathbf{V}^{-1}\mathbf{F})^{-1}\mathbf{F}'\mathbf{V}^{-1}(\mathbf{Y}-\mathbf{f}_0)$$

is

$$E(\widehat{\delta\boldsymbol{\beta}}) = (\mathbf{F}'\mathbf{V}^{-1}\mathbf{F})^{-1}\mathbf{F}'\mathbf{V}^{-1}\Big[\mathbf{F}\delta\boldsymbol{\beta} + \frac{1}{2}\boldsymbol{\kappa}(\delta\boldsymbol{\beta})\Big]$$
$$= \delta\boldsymbol{\beta} + \frac{1}{2}(\mathbf{F}'\mathbf{V}^{-1}\mathbf{F})^{-1}\mathbf{F}'\mathbf{V}^{-1}\boldsymbol{\kappa}(\delta\boldsymbol{\beta}).$$

Under the null hypothesis $H_0 : \mathbf{T}\delta\boldsymbol{\beta} + \frac{1}{2}\boldsymbol{\tau}(\delta\boldsymbol{\beta}) = \mathbf{0}$ it is valid

$$\delta\boldsymbol{\beta} = \mathbf{K}_T\delta\mathbf{u} - \mathbf{T}^{-}\frac{1}{2}\boldsymbol{\tau}(\mathbf{K}_T\delta\mathbf{u}) + \text{terms of higher orders.}$$

Thus

$$\mathbf{T}E(\widehat{\delta\boldsymbol{\beta}}) = \mathbf{T}\Big[\mathbf{K}_T\delta\mathbf{u} - \mathbf{T}^{-}\frac{1}{2}\boldsymbol{\tau}(\mathbf{K}_T\delta\mathbf{u})\Big] + \mathbf{T}\frac{1}{2}(\mathbf{F}'\mathbf{V}^{-1}\mathbf{F})^{-1}\mathbf{F}'\mathbf{V}^{-1}\boldsymbol{\kappa}(\mathbf{K}_T\delta\mathbf{u}) + \dots$$

In the last term the vector $\delta\boldsymbol{\beta}$ is substituted by $\mathbf{K}_T\delta\mathbf{u}$. Since $\mathbf{T}\mathbf{K}_T = \mathbf{0}$ and $\mathbf{T}\mathbf{T}^{-} = \mathbf{I}$, the expression $[E(\widehat{\delta\boldsymbol{\beta}})]'\mathbf{T}'\Big[\mathbf{T}(\mathbf{F}'\mathbf{V}^{-1}\mathbf{F})^{-1}\mathbf{T}'\Big]^{-1}\mathbf{T}E(\widehat{\delta\boldsymbol{\beta}})/\sigma^2 = \Delta$ can be written in the form given in the statement. (Cf. also [1] and [2].) □

Definition 2.3 The quantity

$$K^{(test)}(\boldsymbol{\beta}_0) = \sup\left\{\frac{2\sqrt{\mathbf{b}'\Big\{\mathbf{T}[\mathbf{F}'(\sigma^2\mathbf{V})^{-1}\mathbf{F}]^{-1}\mathbf{T}'\Big\}^{-1}\mathbf{b}}}{\delta\mathbf{u}'\mathbf{K}_T'\mathbf{F}'(\sigma^2\mathbf{V})^{-1}\mathbf{F}\mathbf{K}_T\delta\mathbf{u}} : \delta\mathbf{u}\in R^{k-q}\right\}$$
$$= \sigma\sup\left\{\frac{2\sqrt{\mathbf{b}'\Big\{\mathbf{T}[\mathbf{F}'\mathbf{V}^{-1}\mathbf{F}]^{-1}\mathbf{T}'\Big\}^{-1}\mathbf{b}}}{\delta\mathbf{u}'\mathbf{K}_T'\mathbf{F}'\mathbf{V}^{-1}\mathbf{F}\mathbf{K}_T\delta\mathbf{u}} : \delta\mathbf{u}\in R^{k-q}\right\} = \sigma\mathbf{K}_0^{(test)}(\boldsymbol{\beta}_0),$$

where

$$\mathbf{b} = -\frac{1}{2}\boldsymbol{\tau}(\mathbf{K}_T\delta\mathbf{u}) + \mathbf{T}(\mathbf{F}'\mathbf{V}^{-1}\mathbf{F})^{-1}\mathbf{F}'\mathbf{V}^{-1}\frac{1}{2}\boldsymbol{\kappa}(\mathbf{K}_T\delta\mathbf{u}),$$

is a measure of nonlinearity for test.

Theorem 2.4 *Let δ_{max} be a solution of the equation*

$$P\{\chi_q^2(\delta_{max}) \geq \chi_q^2(0;1-\alpha)\} = \alpha + \varepsilon.$$

Here $\chi_q^2(0;1-\alpha)$ is $(1-\alpha)$-quantile of the chi-square distribution with q degrees of freeedom. Then

$$\delta\boldsymbol{\beta} \in \mathcal{L}_T = \left\{\mathbf{K}_T\delta\mathbf{u} : \delta\mathbf{u}'\mathbf{K}_T'(\mathbf{F}'\mathbf{V}^{-1}\mathbf{F})^{-1}\mathbf{K}_T\delta\mathbf{u} \leq \frac{2\sigma\sqrt{\delta_{max}}}{K_0^{(test)}(\boldsymbol{\beta}_0)}\right\}$$
$$\Rightarrow P_{H_0}\left\{\widehat{\delta\boldsymbol{\beta}}'\mathbf{T}'[\mathbf{T}(\mathbf{F}'\mathbf{V}^{-1}\mathbf{F})^{-1}\mathbf{T}']^{-1}\mathbf{T}\widehat{\delta\boldsymbol{\beta}} \geq \sigma^2\chi_q^2(0;1-\alpha)\right\} \leq \alpha + \varepsilon.$$

Proof In the model (2) the random variable $\widehat{\delta\boldsymbol{\beta}}'\mathbf{T}'\left[\mathbf{T}(\mathbf{F}'\mathbf{V}^{-1}\mathbf{F})^{-1}\mathbf{T}'\right]^{-1}\mathbf{T}\widehat{\delta\boldsymbol{\beta}}$ is distributed as $\sigma^2\chi_q^2(\Delta)$, where Δ is given by Lemma 2.2. With respect to Definition 2.3 we have

$$2\sqrt{\mathbf{b}'\left[\mathbf{T}(\mathbf{F}'\mathbf{V}^{-1}\mathbf{F})^{-1}\mathbf{T}'\right]^{-1}\mathbf{b}} \leq K_0^{(test)}(\boldsymbol{\beta}_0)\delta\mathbf{u}'\mathbf{K}_T'(\mathbf{F}'\mathbf{V}^{-1}\mathbf{F})^{-1}\mathbf{K}_T\delta\mathbf{u}.$$

If

$$K_0^{(test)}(\boldsymbol{\beta}_0)\delta\mathbf{u}'\mathbf{K}_T'(\mathbf{F}'\mathbf{V}^{-1}\mathbf{F})^{-1}\mathbf{K}_T\delta\mathbf{u} \leq 2\sigma\sqrt{\delta_{max}},$$

then

$$2\sqrt{\frac{\mathbf{b}'\left[\mathbf{T}(\mathbf{F}'\mathbf{V}^{-1}\mathbf{F})^{-1}\mathbf{T}'\right]^{-1}\mathbf{b}}{\sigma^2}} = 2\sqrt{\Delta} \leq 2\sqrt{\delta_{max}}$$
$$\Rightarrow P_{H_0}\left\{\chi_q^2(\Delta) \geq \chi_q^2(0;1-\alpha)\right\} \leq \alpha + \varepsilon.$$

Thus

$$\delta\mathbf{u}'\mathbf{K}_T'(\mathbf{F}'\mathbf{V}^{-1}\mathbf{F})^{-1}\mathbf{K}_T\delta\mathbf{u} \leq \frac{2\sigma\sqrt{\delta_{max}}}{K_0^{(test)}(\boldsymbol{\beta}_0)} \Rightarrow P_{H_0}\left\{\chi_q^2(\Delta) \geq \chi_q^2(0;1-\alpha)\right\} \leq \alpha+\varepsilon.$$

□

Remark 2.5 If $\mathbf{T}\delta\boldsymbol{\beta} + \frac{1}{2}\boldsymbol{\tau}(\delta\boldsymbol{\beta}) \neq \mathbf{0}$, i.e. the null hypothesis is not true, then

$$\delta = \left[\mathbf{T}\delta\boldsymbol{\beta} + \frac{1}{2}\mathbf{T}\mathbf{C}_0^{-1}\mathbf{F}'\mathbf{V}^{-1}\boldsymbol{\kappa}(\delta\boldsymbol{\beta})\right]'(\mathbf{T}\mathbf{C}_0\mathbf{T}')^{-1}\left[\mathbf{T}\delta\boldsymbol{\beta} + \frac{1}{2}\mathbf{T}\mathbf{C}_0^{-1}\mathbf{F}'\mathbf{V}^{-1}\boldsymbol{\kappa}(\delta\boldsymbol{\beta})\right],$$

where $\mathbf{C}_0 = \mathbf{F}'\mathbf{V}^{-1}\mathbf{F}$. Thus at the alternative hypothesis the power function

$$p(\delta\boldsymbol{\beta}) = P_{H_a}\left\{\chi_t^2(\delta) \geq \chi_t^2(0;1-\alpha)\right\}$$

has a different values at points $\delta\boldsymbol{\beta}$ and $-\delta\boldsymbol{\beta}$, respectively, opposite to the case of the null hypothesis where these values are identical. It makes an investigation of a linearization region for the power function more complicated than it is at the null hypothesis.

Lemma 2.6 *The* $(1-\alpha)$*-confidence ellipsoid for the parameter* $\boldsymbol{\beta}$ *in the model (1) is*

$$\mathcal{E} = \{\mathbf{u} : (\mathbf{u} - \widehat{\delta\boldsymbol{\beta}})'\mathbf{F}'\mathbf{V}^{-1}\mathbf{F}(\mathbf{u} - \widehat{\delta\boldsymbol{\beta}}) \leq \sigma^2\chi_k^2(0; 1-\alpha)\}.$$

Proof Cf. [4], chpt. 4. □

Remark 2.7 If

$$\sigma^2\chi_q^2(0; 1-\alpha) \ll \frac{2\sigma\sqrt{\delta_{max}}}{K_0^{(test)}(\boldsymbol{\beta}_0)},$$

then the model (2) can be substituted by (1) when the test of hypothesis is performed. Thus the value of σ must satisfy the strong inequality $\sigma \ll \sigma_{crit}$, where

$$\sigma_{crit} = \frac{2\sqrt{\delta_{max}}}{\chi_q^2(0; 1-\alpha)K_0^{test}(\boldsymbol{\beta}_0)}.$$

3 Numerical example

Let a class of regression function be $\{f(x) = \beta_1\exp(-\beta_2 x) : \beta_1, \beta_2 \in R^1\}$. The null hypothesis states that all these functions attain the same value equal to 1 at the point $x = 10$ (cf. also [3]).

The measurement is realized at the points $x_i \in \{1, 3, 5, 7, 9\}$. Thus

$$H_0 : \ln\beta_1 - 10\beta_2 = 0, \quad H_a : \ln\beta_1 - 10\beta_2 \neq 0.$$

The regression model is

$$\mathbf{Y} \sim N_5[\mathbf{f}(\boldsymbol{\beta}), \sigma^2\mathbf{I}], \quad \boldsymbol{\beta} \in R^2,$$

where

$$\{\mathbf{f}(\boldsymbol{\beta})\}_i = \beta_1\exp(-\beta_2 x_i), \quad i = 1, 2, 3, 4, 5,$$

$$t(\boldsymbol{\beta}) = \ln\beta_1 - 10\beta_2 = 0,$$

$$\{\mathbf{F}\}_{i,\cdot} = \left(\exp(-\beta_2^{(0)}x_i), -\beta_1 x_i\exp(-\beta_2^{(0)}x_i)\right), \quad i = 1, \ldots, 5,$$

$$\mathbf{F}_i = \begin{pmatrix} 0, & -x_i\exp(-\beta_2^{(0)}x_i) \\ -x_i\exp(-\beta_2^{(0)}x_i), & \beta_1^{(0)}x_i^2\exp(-\beta_2^{(0)}x_i) \end{pmatrix}, \quad i = 1, \ldots, 5,$$

$$\mathbf{T} = \left(\frac{1}{\beta_1^{(0)}}, -10\right), \quad \mathbf{K}_T = \begin{pmatrix} \beta_1^{(0)} \\ 0.1 \end{pmatrix},$$

$$\kappa_i(\mathbf{K}_T\delta u) = (\delta u)^2\left(-0.2x_i\beta_1^{(0)}\exp(-\beta_2^{(0)}x_i) + 0.01\beta_1^{(0)}x_i^2\exp(-\beta_2^{(0)}x_i)\right), \quad i = 1, \ldots, 5,$$

$$\mathbf{F}'\mathbf{F} = \begin{pmatrix} \sum_{i=1}^5\exp(-2\beta_2^{(0)}x_i), & -\sum_{i=1}^5\beta_1^{(0)}x_i\exp(-2\beta_2^{(0)}x_i) \\ -\sum_{i=1}^5\beta_1^{(0)}x_i\exp(-2\beta_2^{(0)}x_i), & \sum_{i=1}^5(\beta_1^{(0)})^2x_i^2(\exp(-2\beta_2^{(0)}x_i) \end{pmatrix},$$

$$\mathbf{b} = -\frac{1}{2}\tau(\mathbf{K}_T\delta u) + \mathbf{T}(\mathbf{F}'\mathbf{F})^{-1}\mathbf{F}'\frac{1}{2}\boldsymbol{\kappa}(\mathbf{K}_T\delta u) = \frac{1}{2}(1+A)(\delta u)^2,$$

where

$$A = \left(\frac{1}{\beta_1^{(0)}}, -10\right)(\mathbf{F}'\mathbf{F})^{-1}\mathbf{F}' \times$$
$$\times \begin{pmatrix} \vdots \\ -0.2x_i\beta_1^{(0)}\exp(-\beta_2^{(0)}x_i) + 0.01\beta_1^{(0)}x_i^2\exp(-\beta_2^{(0)}x_i) \\ \vdots \end{pmatrix}.$$

Further

$$K^{(test)}(\boldsymbol{\beta}_0) = \sigma \frac{\sqrt{(1+A)\left[(1/\beta_1^{(0)}, -10)(\mathbf{F}'\mathbf{F})^{-1}\begin{pmatrix}1/\beta_1^{(0)}\\ -10\end{pmatrix}\right]^{-1}(1+A)}}{(\beta_1^{(0)}, 0.1)\mathbf{F}'\mathbf{F}\begin{pmatrix}\beta_1^{(0)}\\ 0.1\end{pmatrix}} = \sigma K_0^{(test)},$$

$$K_0^{(test)} = \frac{|1+A|}{(\beta_1^{(0)}, 0.1)\mathbf{F}'\mathbf{F}\begin{pmatrix}\beta_1^{(0)}\\ 0.1\end{pmatrix}\sqrt{(1/\beta_1^{(0)}, -10)(\mathbf{F}'\mathbf{F})^{-1}\begin{pmatrix}1/\beta_1^{(0)}\\ -10\end{pmatrix}}}.$$

$$P\{\chi_1^2(\delta_{max}) \geq \chi_1^2(0; 0.95)\} = 0.05 + 0.05 \Rightarrow \delta_{max} = 0.426, \quad \chi_1^2(0; 0.95) = 3.84,$$

$$\sigma_{crit} = \frac{2\sqrt{0.451}}{3.84K_0^{(test)}(\boldsymbol{\beta}_0)} = \frac{0.349774}{K_0^{(test)}(\boldsymbol{\beta}_0)}.$$

Some numerical values were obtained by the help of [5] and they are given in the following table.

Table 1

$\boldsymbol{\beta}^{(0)}$	$\binom{0.1}{-0.230}$	$\binom{0.2}{-0.161}$	$\binom{0.3}{-0.120}$	$\binom{0.5}{-0.069}$
$K_0^{(test)}(\boldsymbol{\beta}_0)$	0.613	0.406	0.306	0.206
σ_{crit}	0.554	0.837	1.110	1.649
$\boldsymbol{\beta}^{(0)}$	$\binom{1}{0}$	$\binom{5}{0.161}$	$\binom{10}{0.230}$	$\binom{15}{0.271}$
$K_0^{(test)}(\boldsymbol{\beta}_0)$	0.113	0.024	0.012	0.008
σ_{crit}	3.01	14.15 2	28.31	42.47

If the value of σ in the actual experiment is smaller than σ_{crit} from Table 1, then the theory of linear regression model can be used when the test of hypothesis is performed.

It is advisable to notice a strong dependence of the quantities $K_0^{(test)}$ and σ_{crit}, respectively, on the vector $\boldsymbol{\beta}^{(0)}$.

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References

[1] Kubáček, L., Kubáčková, L.: *Regression models with a weak nonlinearity.* Technical report Nr. 1998.1, Universität Stuttgart, 1998, 1–67.

[2] Kubáček, L., Kubáčková, L.: Statistics and Metrology. *Vyd. Univ. Palackého, Olomouc*, 2000 (in Czech).

[3] Kubáček, L., Kubáčková, L.: *Statistical problems of a determination of isobestic points.* Folia Fac. Sci. Nat. Univ. Masarykianae Brunensis, Math. **11** (2002), 139–150.

[4] Rao, C. R.: Linear Statistical Inference and its Application. *J. Wiley, New York–London–Sydney*, 1965.

[5] Tesaříková, E., Kubáček, L.: *A test in nonlinear regression models. Demoprogram.* Department of Algebra and Geometry, Faculta of Science, Palacký University, Olomouc, 2004 (in Czech).

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One Singular Multivariate Linear Model with Nuisance Parameters

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Abstract

The multivariate linear model, in which the matrix of the first order parameters is divided into two matrices: to the matrix of the useful parameters and to the matrix of the nuisance parameters, is considered.

Key words: Singular multivariate linear model, useful and nuisance parameters, BLUE.

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1 Introduction

There are two approaches in the problem of nuisance parameters in the linear models of various structures.

The first one respects the structure of the model and seeks to find classes of linear functionals of useful (main) parameters such that their estimators allow the nuisance parameters to be neglected; the estimators computed under disregarding nuisance parameters remain to be unbiased and efficient. The variance of the estimator belonging to the abovementioned class could behave analogously. The determination of the class having such attributes is of a great importance in practice because the number of nuisance parameters in real situations can be greater than the number of useful parameters.

The second approach solves the problem of nuisance parameters by their elimination by a transformation of the observation vector provided this transformation is not allowed to cause a loss of information on the useful parameters (see [7]).

The aim of this paper is to apply the first approach to one of the multivariate models.

2 Notations and auxiliary statements

Let R^n denote the space of all n-dimensional real vectors, let $\mathbf{u}_p$ and $\mathbf{A}_{m,n}$ denote a real column p-dimensional vector and a real $m \times n$ matrix, respectively. The symbols $\mathbf{A}', \mathbf{A}^{(j)}, \mathscr{M}(\mathbf{A}), \mathscr{N}(\mathbf{A}), r(\mathbf{A}), Tr(\mathbf{A})$ will denote transpose, j-th column, range, null space, rank and trace of the matrix $\mathbf{A}$, respectively. Further $vec(\mathbf{A})$ will denote the column vector $((\mathbf{A}^{(1)})', \dots, (\mathbf{A}^{(n)})')'$ created by the columns of the matrix $\mathbf{A}$. The symbol $\mathbf{A} \otimes \mathbf{B}$ will denote the Kronecker (tensor) product of the matrices $\mathbf{A}, \mathbf{B}$; $\mathbf{A}^-$ will denote an arbitrary generalized inverse of $\mathbf{A}$ (satisfying $\mathbf{A}\mathbf{A}^-\mathbf{A} = \mathbf{A}$), $\mathbf{A}^+$ will denote a Moore–Penrose generalized inverse of the matrix $\mathbf{A}$ (satisfying $\mathbf{A}\mathbf{A}^+\mathbf{A} = \mathbf{A}$, $\mathbf{A}^+\mathbf{A}\mathbf{A}^+ = \mathbf{A}^+$, $(\mathbf{A}\mathbf{A}^+)' = \mathbf{A}\mathbf{A}^+$, $(\mathbf{A}^+\mathbf{A})' = \mathbf{A}^+\mathbf{A}$). Moreover $\mathbf{P}_A$ and $\mathbf{M}_A = \mathbf{I} - \mathbf{P}_A$ will stand for the ortogonal projector onto $\mathscr{M}(\mathbf{A})$ and $\mathscr{M}^{\perp}(\mathbf{A}) = \mathscr{N}(\mathbf{A}')$, respectively. The symbol $\mathbf{I}$ denotes the identity matrix, $\mathbf{O}_{m,n}$ the $m \times n$ null matrix, $\mathbf{o}$ the null element. We write

$$\mathbf{A} \overset{\leq}{\scriptstyle L} \mathbf{B} \iff \mathbf{B} - \mathbf{A} \text{ is p.s.d.}$$

If $\mathscr{M}(\mathbf{A}) \subset \mathscr{M}(\mathbf{V})$, $\mathbf{V}$ p.s.d., then the symbol $\mathbf{P}_A^V$ denotes the projector on the subspace $\mathscr{M}(\mathbf{A})$ in the $\mathbf{V}$-seminorm given by the matrix $\mathbf{V}$,

$$||\mathbf{x}||_V = \sqrt{\mathbf{x}'\mathbf{V}\mathbf{x}}; \qquad \mathbf{M}_A^V = \mathbf{I} - \mathbf{P}_A^V = \mathbf{I} - \mathbf{A}(\mathbf{A}'\mathbf{V}\mathbf{A})^-\mathbf{A}'\mathbf{V}.$$

Let $\mathbf{N}_{n,n}$ is p.d. (p.s.d.) matrix and $\mathbf{A}_{m,n}$ an arbitrary matrix, then the symbol $\mathbf{A}^-_{m(N)}$ denotes the matrix satisfying

$$\mathbf{A}\mathbf{A}^-_{m(N)}\mathbf{A} = \mathbf{A} \quad \text{and} \quad \mathbf{N}\mathbf{A}^-_{m(N)}\mathbf{A} = [\mathbf{N}\mathbf{A}^-_{m(N)}\mathbf{A}]'.$$

($\mathbf{A}^-_{m(N)}\mathbf{y}$ is a solution of the consistent system $\mathbf{A}\mathbf{x} = \mathbf{y}$ whose N-seminorm is minimal, see [4], p.151). $\mathbf{A}^-_{m(N)}$ is called a minimum N-seminorm g-inverse of the matrix $\mathbf{A}$. Let $\mathscr{A}^-_{m(N)}$ be a class of all matrices $\mathbf{A}^-_{m(N)}$.

Assertion 1 (see [1], Lemma 10.1.18)

$$\mathscr{M}(\mathbf{A}') \subset \mathscr{M}(\mathbf{N}) \implies \mathbf{N}^-\mathbf{A}'(\mathbf{A}\mathbf{N}^-\mathbf{A}')^- \in \mathscr{A}^-_{m(N)},$$

otherwise

$$(\mathbf{N} + \mathbf{A}'\mathbf{A})^-\mathbf{A}'[\mathbf{A}(\mathbf{N} + \mathbf{A}'\mathbf{A})^-\mathbf{A}']^- \in \mathscr{A}^-_{m(N)}.$$

Assertion 2 (see [1], Lemma 10.1.35) Let $\mathbf{S}$ be any $n \times k$ matrix and $\mathbf{N}$ an $n \times n$ p.s.d. matrix.

1. If $\mathbf{N}$ is p.d., then $(\mathbf{M}_S\mathbf{N}\mathbf{M}_S)^+ = \mathbf{N}^{-1} - \mathbf{N}^{-1}\mathbf{S}(\mathbf{S}'\mathbf{N}^{-1}\mathbf{S})^-\mathbf{S}'\mathbf{N}^{-1}$.
2. If $\mathbf{N}$ is not p.d., however $\mathscr{M}(\mathbf{S}) \subset \mathscr{M}(\mathbf{N})$, then
$$(\mathbf{M}_S\mathbf{N}\mathbf{M}_S)^+ = \mathbf{N}^+ - \mathbf{N}^+\mathbf{S}(\mathbf{S}'\mathbf{N}^-\mathbf{S})^-\mathbf{S}'\mathbf{N}^+.$$
3. In general case
$$(\mathbf{M}_S\mathbf{N}\mathbf{M}_S)^+ = (\mathbf{N}+\mathbf{S}\mathbf{S}')^+ - (\mathbf{N}+\mathbf{S}\mathbf{S}')^+\mathbf{S}[\mathbf{S}'(\mathbf{N}+\mathbf{S}\mathbf{S}')^-\mathbf{S}]^-\mathbf{S}'(\mathbf{N}+\mathbf{S}\mathbf{S}')^+.$$
4. $$(\mathbf{M}_S\mathbf{N}\mathbf{M}_S)^+ = (\mathbf{M}_S\mathbf{N}\mathbf{M}_S)^+\mathbf{M}_S = \mathbf{M}_S(\mathbf{M}_S\mathbf{N}\mathbf{M}_S)^+ = \mathbf{M}_S(\mathbf{M}_S\mathbf{N}\mathbf{M}_S)^+\mathbf{M}_S.$$

Assertion 3 (see [2], Lemma 7, p. 65)

$$\mathscr{M}(\mathbf{B}) \subset \mathscr{M}(\mathbf{A}) \iff \mathbf{A}\mathbf{A}^-\mathbf{B} = \mathbf{B},$$
$$\mathscr{M}(\mathbf{B}') \subset \mathscr{M}(\mathbf{A}') \iff \mathbf{B}\mathbf{A}^-\mathbf{A} = \mathbf{B}.$$

Assertion 4 (see [2], Lemma 8, p. 65)

$$\mathbf{A}\mathbf{B}^-\mathbf{C} \text{ is invariant to the choice of the g-inverse } \mathbf{B}^-$$
$$\iff \mathscr{M}(\mathbf{A}') \subset \mathscr{M}(\mathbf{B}') \text{ and } \mathscr{M}(\mathbf{C}) \subset \mathscr{M}(\mathbf{B}).$$

Assertion 5 If $\mathbf{N}$ is p.s.d. and $\mathbf{A}$ such matrices that $\mathscr{M}(\mathbf{A}) \subset \mathscr{M}(\mathbf{N})$, then

$$\mathscr{M}(\mathbf{A}') = \mathscr{M}(\mathbf{A}'\mathbf{N}^-\mathbf{A}).$$

Proof $\mathbf{A}'\mathbf{N}^-\mathbf{A}$ is invariant to the choice of g-inverse. As $\mathscr{M}(\mathbf{A}'\mathbf{N}^-\mathbf{A}) \subset \mathscr{M}(\mathbf{A}')$, it is sufficient to prove, that $r(\mathbf{A}'\mathbf{N}^+\mathbf{A}) = r(\mathbf{A}')$. Let $\mathbf{N}^+ = \mathbf{J}\mathbf{J}'$, then $r(\mathbf{A}'\mathbf{N}^+\mathbf{A}) = r(\mathbf{A}'\mathbf{J})$. There exists a matrix $\mathbf{F}$ such that $\mathbf{A} = \mathbf{N}\mathbf{F}$. Thus $r(\mathbf{A}') = r(\mathbf{F}'\mathbf{N}) = r(\mathbf{F}'\mathbf{N}\mathbf{N}^+\mathbf{N}) = r(\mathbf{A}'\mathbf{N}^+\mathbf{N}) \le r(\mathbf{A}'\mathbf{N}^+) \le r(\mathbf{A}'\mathbf{J}) \le r(\mathbf{A}')$. □

3 Singular multivariate linear regression model

Let
$$\mathbf{Y} = \mathbf{X}_1\mathbf{B}_1\mathbf{Z}_1 + \mathbf{X}_2\mathbf{B}_2\mathbf{Z}_2 + \boldsymbol{\varepsilon}, \tag{1}$$
be a multivariate linear model under consideration.

Here $\mathbf{Y}$ is an $n \times m$ observation matrix, $\mathbf{X}_1$ of the type $n \times k$, $\mathbf{Z}_1$ of the type $r \times m$, $\mathbf{X}_2$ of the type $n \times l$, $\mathbf{Z}_2$ of the type $s \times m$ are known nonzero matrices.

$\mathbf{B}_1$ of the type $k \times r$ and $\mathbf{B}_2$ of the type $l \times s$ are matrices of unknown nonrandom parameters and $\boldsymbol{\varepsilon}$ of the type $n \times m$ is a random matrix.

Let us consider the situation, where $\mathbf{B}_1$ is a matrix of useful parameters which (or their functions) have to be estimated from the observation matrix and $\mathbf{B}_2$ is a matrix of nuisance parameters.

As it was already said the purpose of this paper is to characterize the class of all linear functions of the useful parameters $vec(\mathbf{B})$ which are unbiasedly estimable under the model with nuisance parameters and under the model, where the nuisance parameters are neglected and estimators of which have the same variance in both models mentioned.

A parametric function $\mathbf{p}'vec(\mathbf{B}_1)$ is said to be unbiasedly estimable under the model (1) if there exists an estimator $\mathbf{f}'vec(\mathbf{Y}), \mathbf{f} \in R^{mn}$, such that $E[\mathbf{f}'vec(\mathbf{Y})] = \mathbf{p}'vec(\mathbf{B}_1)$, $\forall vec(\mathbf{B}_1)$, $\forall vec(\mathbf{B}_2)$.

Lemma 1 *The model (1) can be equivalently written in the form*

$$vec(\mathbf{Y}) = [\mathbf{Z}_1' \otimes \mathbf{X}_1, \mathbf{Z}_2' \otimes \mathbf{X}_2] \begin{pmatrix} vec(\mathbf{B}_1) \\ vec(\mathbf{B}_2) \end{pmatrix} + vec(\boldsymbol{\varepsilon}).$$

Proof The assertion is a consequence of

$$vec(\mathbf{ABC}) = (\mathbf{C}' \otimes \mathbf{A})vec(\mathbf{B}),$$

valid for all matrices of corresponding types. □

Suppose that the observation vector $vec(\mathbf{Y})$ has the mean value

$$E(vec(\mathbf{Y})) = [\mathbf{Z}_1' \otimes \mathbf{X}_1, \mathbf{Z}_2' \otimes \mathbf{X}_2] \begin{pmatrix} vec(\mathbf{B}_1) \\ vec(\mathbf{B}_2) \end{pmatrix},$$

and that the columns of the observation matrix $\mathbf{Y}$ satisfy

$$cov(\mathbf{Y}^{(i)}, \mathbf{Y}^{(j)}) = \mathbf{O},\ \forall i \neq j, \quad var[\mathbf{Y}^{(j)}] = \boldsymbol{\Sigma},\ \forall j = 1, \ldots, m,$$

where $\boldsymbol{\Sigma}$ is at least positive semidefinite known matrix. Thus

$$var[vec(\mathbf{Y})] = \mathbf{I}_{m,m} \otimes \boldsymbol{\Sigma}_{n,n}.$$

We consider the linear model

$$\left[vec(\mathbf{Y}), (\mathbf{Z}_1' \otimes \mathbf{X}_1, \mathbf{Z}_2' \otimes \mathbf{X}_2) \begin{pmatrix} vec(\mathbf{B}_1) \\ vec(\mathbf{B}_2) \end{pmatrix}, \mathbf{I} \otimes \boldsymbol{\Sigma}\right], \tag{2}$$

with nuisance parameters (great model) and the linear model

$$\left[vec(\mathbf{Y}), (\mathbf{Z}_1' \otimes \mathbf{X}_1)vec(\mathbf{B}_1), \mathbf{I} \otimes \boldsymbol{\Sigma}\right], \tag{3}$$

where nuisance parameters are neglected (small model).

The paper [5] deals with following assumption

$$\mathscr{M}(\mathbf{Z}_1' \otimes \mathbf{X}_1, \mathbf{Z}_2' \otimes \mathbf{X}_2) \subset \mathscr{M}(\mathbf{I} \otimes \boldsymbol{\Sigma}). \tag{4}$$

Here the general situation will be considered.

Notation 2 Let $\mathscr{E}_a$ and $\mathscr{E}$ denote the sets of all linear functions of $vec(\mathbf{B}_1)$ which are unbiasedly estimable under the model (2) and (3), respectively (see [8]). The index a will indicate, that the estimator is considered in the complete model, i.e. in the model with nuisance parameters.

Lemma 2

$$\mathscr{E} = \{\mathbf{p}'vec(\mathbf{B}) : \mathbf{p} \in \mathscr{M}(\mathbf{Z}_1 \otimes \mathbf{X}_1')\}. \tag{5}$$

$$\begin{aligned}\mathscr{E}_a &= \{\mathbf{p}'vec(\mathbf{B}) : \mathbf{p} \in \mathscr{M}[(\mathbf{Z}_1 \otimes \mathbf{X}_1')\mathbf{M}_{Z_2'\otimes X_2}] \\ &= \mathscr{M}[(\mathbf{Z}_1 \otimes \mathbf{X}_1') - (\mathbf{Z}_1\mathbf{P}_{Z_2'} \otimes \mathbf{X}_1'\mathbf{P}_{X_2})]\}.\end{aligned} \tag{6}$$

Proof see [5], Lemma 2.

Comparing (5) and (6) it is obvious that

$$\mathscr{E}_a \subset \mathscr{E}.$$

Moreover,

Lemma 3 *Under the condition* $\mathscr{E}_a \subset \mathscr{E}$

$$\mathscr{E}_a = \mathscr{E} \iff \mathscr{M}(\mathbf{Z}_1' \otimes \mathbf{X}_1) \cap \mathscr{M}(\mathbf{Z}_2' \otimes \mathbf{X}_2) = \{\mathbf{o}\} \tag{7}$$

Proof see [5], Lemma 3.

We assume throughout that $\mathscr{M}(\mathbf{Z}_1' \otimes \mathbf{X}_1) \not\subset \mathscr{M}(\mathbf{Z}_2' \otimes \mathbf{X}_2)$. If $\mathscr{M}(\mathbf{Z}_1' \otimes \mathbf{X}_1) \subset \mathscr{M}(\mathbf{Z}_2' \otimes \mathbf{X}_2)$, then $\mathscr{M}[(\mathbf{Z}_1 \otimes \mathbf{X}_1') - (\mathbf{Z}_1\mathbf{P}_{Z_2'} \otimes \mathbf{X}_1'\mathbf{P}_{X_2})] = \{\mathbf{o}\}$.

Notation 3 Let us denote

$$\mathbf{T} = (\mathbf{I} \otimes \mathbf{\Sigma}) + (\mathbf{Z}_1' \otimes \mathbf{X}_1)(\mathbf{Z}_1 \otimes \mathbf{X}_1') = (\mathbf{I} \otimes \mathbf{\Sigma}) + (\mathbf{Z}_1'\mathbf{Z}_1 \otimes \mathbf{X}_1\mathbf{X}_1').$$

Theorem 1 *The BLUE of the vector function* $(\mathbf{Z}_1' \otimes \mathbf{X}_1)vec(\mathbf{B}_1)$ *under the model (3) is given by*

$$(\mathbf{Z}_1' \widehat{\otimes \mathbf{X}_1)vec(}\mathbf{B}_1) = \mathbf{P}^{T^+}_{Z_1'\otimes X_1} vec(\mathbf{Y}), \tag{8}$$

$$\begin{aligned}&var[(\mathbf{Z}_1' \widehat{\otimes \mathbf{X}_1)vec(}\mathbf{B}_1)] = \\ &= (\mathbf{Z}_1' \otimes \mathbf{X}_1)\left\{\left[(\mathbf{Z}_1 \otimes \mathbf{X}_1')\mathbf{T}^+(\mathbf{Z}_1' \otimes \mathbf{X}_1)\right]^- - \mathbf{I}\right\}(\mathbf{Z}_1 \otimes \mathbf{X}_1').\end{aligned} \tag{9}$$

Proof According to Theorem 3.1.3 in [1]

$$(\mathbf{Z}_1' \widehat{\otimes \mathbf{X}_1)vec(}\mathbf{B}_1) = (\mathbf{Z}_1' \otimes \mathbf{X}_1)\left\{[(\mathbf{Z}_1' \otimes \mathbf{X}_1)'_{m(I\otimes\Sigma)}]^-\right\}' vec(\mathbf{Y})$$

$$= (\mathbf{Z}_1' \otimes \mathbf{X}_1)[(\mathbf{Z}_1 \otimes \mathbf{X}_1')\mathbf{T}^+(\mathbf{Z}_1' \otimes \mathbf{X}_1)]^-(\mathbf{Z}_1 \otimes \mathbf{X}_1')\mathbf{T}^+ vec(\mathbf{Y}) = \mathbf{P}^{T^+}_{Z_1'\otimes X} vec(\mathbf{Y}),$$

where Assertion 1, the inclusion $\mathscr{M}(\mathbf{Z}_1' \otimes \mathbf{X}_1) \subset \mathscr{M}(\mathbf{T})$ and the fact that under the model (3)

$$P[vec(\mathbf{Y}) \in \mathscr{M}(\mathbf{Z}_1' \otimes \mathbf{X}_1, \mathbf{I} \otimes \mathbf{\Sigma})] = 1$$

have been utilized. Further

$$\begin{aligned}
var[\widehat{(\mathbf{Z}_1'\otimes\mathbf{X}_1)vec(}\mathbf{B}_1)] &= (\mathbf{Z}_1'\otimes\mathbf{X}_1)\left[(\mathbf{Z}_1\otimes\mathbf{X}_1')\mathbf{T}^+(\mathbf{Z}_1'\otimes\mathbf{X}_1)\right]^-(\mathbf{Z}_1\otimes\mathbf{X}_1')\mathbf{T}^+\\
&\times[\mathbf{T}-(\mathbf{Z}_1'\mathbf{Z}_1\otimes\mathbf{X}_1\mathbf{X}_1')]\mathbf{T}^+(\mathbf{Z}_1'\otimes\mathbf{X}_1)[(\mathbf{Z}_1\otimes\mathbf{X}_1')\mathbf{T}^+(\mathbf{Z}_1'\otimes\mathbf{X}_1)]^-(\mathbf{Z}_1\otimes\mathbf{X}_1')\\
&= (\mathbf{Z}_1'\otimes\mathbf{X}_1)\left[(\mathbf{Z}_1\otimes\mathbf{X}_1')\mathbf{T}^+(\mathbf{Z}_1'\otimes\mathbf{X}_1)\right]^-(\mathbf{Z}_1\otimes\mathbf{X}_1')\mathbf{T}^+\mathbf{T}\mathbf{T}^+(\mathbf{Z}_1'\otimes\mathbf{X}_1)\\
&\times\left[(\mathbf{Z}_1\otimes\mathbf{X}_1')\mathbf{T}^+(\mathbf{Z}_1'\otimes\mathbf{X}_1)\right]^-(\mathbf{Z}_1\otimes\mathbf{X}_1')\\
&-(\mathbf{Z}_1'\otimes\mathbf{X}_1)\left[(\mathbf{Z}_1\otimes\mathbf{X}_1')\mathbf{T}^+(\mathbf{Z}_1'\otimes\mathbf{X}_1)\right]^-(\mathbf{Z}_1\otimes\mathbf{X}_1')\mathbf{T}^+(\mathbf{Z}_1'\otimes\mathbf{X}_1)\\
&\times(\mathbf{Z}_1\otimes\mathbf{X}_1')\mathbf{T}^+(\mathbf{Z}_1'\otimes\mathbf{X}_1)\left[(\mathbf{Z}_1\otimes\mathbf{X}_1')\mathbf{T}^+(\mathbf{Z}_1'\otimes\mathbf{X}_1)\right]^-(\mathbf{Z}_1\otimes\mathbf{X}_1')\\
&= (\mathbf{Z}_1'\otimes\mathbf{X}_1)\left\{\left[(\mathbf{Z}_1\otimes\mathbf{X}_1')\mathbf{T}^+(\mathbf{Z}_1'\otimes\mathbf{X}_1)\right]^- - \mathbf{I}\right\}(\mathbf{Z}_1\otimes\mathbf{X}_1').
\end{aligned}$$

The Assertion 3, the equality $\mathscr{M}[\mathbf{Z}_1'\otimes\mathbf{X}_1] = \mathscr{M}[(\mathbf{Z}_1'\otimes\mathbf{X}_1)\mathbf{T}^+(\mathbf{Z}_1\otimes\mathbf{X}_1')]$ and the fact, that under the model (3) $P[vec(\mathbf{Y})\in\mathscr{M}(\mathbf{Z}_1'\otimes\mathbf{X}_1,\mathbf{I}\otimes\mathbf{\Sigma})] = 1$ have been taken into account. □

Theorem 2 *Let us assume that* $\mathscr{M}(\mathbf{Z}_2'\otimes\mathbf{X}_2)\subset\mathscr{M}(\mathbf{M}_{Z_1'\otimes X_1})$*, then the BLUE of the parametric function* $\mathbf{p}'vec(\mathbf{B}_1)$*,* $\mathbf{p}\in\mathscr{M}[(\mathbf{Z}_1\otimes\mathbf{X}_1')\mathbf{M}_{Z_2'\otimes X_2}]$ *in the model (2) is of the form* $\mathbf{g}'vec(\mathbf{Y})$ *where*

$$\begin{aligned}
\mathbf{g} &= \left[\mathbf{M}_{Z_2'\otimes X_2}^{M_{Z_1'\otimes X_1}}(\mathbf{I}\otimes\mathbf{\Sigma}) + (\mathbf{Z}_1'\mathbf{Z}_1\otimes\mathbf{X}_1\mathbf{X}_1')\right]^-(\mathbf{Z}_1'\otimes\mathbf{X}_1)\\
&\times\left\{(\mathbf{Z}_1\otimes\mathbf{X}_1')\left[\mathbf{M}_{Z_2'\otimes X_2}^{M_{Z_1'\otimes X_1}}(\mathbf{I}\otimes\mathbf{\Sigma}) + (\mathbf{Z}_1'\mathbf{Z}_1\otimes\mathbf{X}_1\mathbf{X}_1')\right]^-(\mathbf{Z}_1'\otimes\mathbf{X}_1)\right\}^-\mathbf{p}.
\end{aligned}$$

Proof Let us denote $\mathscr{U}_0$ the class of all unbiased estimators of the null function $\mathbf{p}'vec(\mathbf{B}_1) = 0$, i.e.

$$\begin{aligned}
\mathscr{U}_0 = \{\mathbf{g}_0'vec(\mathbf{Y}) : E[\mathbf{g}_0'vec(\mathbf{Y})] &= \mathbf{g}_0'[(\mathbf{Z}_1'\otimes\mathbf{X}_1)vec(\mathbf{B}_1) + (\mathbf{Z}_2'\otimes\mathbf{X}_2)vec(\mathbf{B}_2)]\\
&= \mathbf{p}'vec(\mathbf{B}_1) = 0,\ \forall vec(\mathbf{B}_1),\ \forall vec(\mathbf{B}_2)\}\\
&= \{\mathbf{u}'\mathbf{M}_{(Z_1'\otimes X_1, Z_2'\otimes X_2)}vec(\mathbf{Y}) : \mathbf{u}\in R^{rk+sl}\}.
\end{aligned}$$

According to the basic lemma on the best estimators (see [3], p. 84) the statistic $\mathbf{g}'vec(\mathbf{Y})$ is the BLUE of the function $\mathbf{p}'vec(\mathbf{B}_1)$ iff

$$\begin{aligned}
&cov[\mathbf{u}'\mathbf{M}_{(Z_1'\otimes X_1, Z_2'\otimes X_2)}vec(\mathbf{Y}), \mathbf{g}'vec(\mathbf{Y})] =\\
&= \mathbf{u}'\mathbf{M}_{(Z_1'\otimes X_1, Z_2'\otimes X_2)}(\mathbf{I}\otimes\mathbf{\Sigma})\mathbf{g} = 0,\quad \forall\mathbf{u}\in R^{rk+sl},\\
&\iff \mathbf{M}_{(Z_1'\otimes X_1, Z_2'\otimes X_2)}(\mathbf{I}\otimes\mathbf{\Sigma})\mathbf{g} = \mathbf{o}.
\end{aligned}$$

Thus we have to find a vector $\mathbf{g}$ such that

$$\mathbf{M}_{(Z_1'\otimes X_1, Z_2'\otimes X_2)}(\mathbf{I}\otimes\mathbf{\Sigma})\mathbf{g} = \mathbf{o}\wedge(\mathbf{Z}_1\otimes\mathbf{X}_1')\mathbf{g} = \mathbf{p}.$$

Using the relation (see [6], Lemma 1)

$$\mathbf{M}_{(Z_1'\otimes X_1, Z_2'\otimes X_2)} = \mathbf{M}_{Z_1'\otimes X_1}\mathbf{M}_{Z_2'\otimes X_2}^{M_{Z_1'\otimes X_1}},$$

and notation $\mathbf{A} = \mathbf{Z}_1' \otimes \mathbf{X}_1,\ \mathbf{B} = \mathbf{Z}_2' \otimes \mathbf{X}_2$ we get

$$\mathbf{P}_A\mathbf{M}_A^{M_B}(\mathbf{I}\otimes\mathbf{\Sigma})\mathbf{g} + \mathbf{M}_A\mathbf{M}_A^{M_B}(\mathbf{I}\otimes\mathbf{\Sigma})\mathbf{g} = \mathbf{M}_A^{M_B}(\mathbf{I}\otimes\mathbf{\Sigma})\mathbf{g},$$

it means we must find the vector $\mathbf{g}$ such that

$$(\mathbf{Z}_1'\otimes\mathbf{X}_1)(\mathbf{A}'\mathbf{A})^-\mathbf{A}'\mathbf{M}_A^{M_B}(\mathbf{I}\otimes\mathbf{\Sigma})\mathbf{g} = \mathbf{M}_A^{M_B}(\mathbf{I}\otimes\mathbf{\Sigma})\mathbf{g} \wedge (\mathbf{Z}_1\otimes\mathbf{X}_1')\mathbf{g} = \mathbf{p},$$

i.e. vector $\mathbf{g}$ such that

$$(\mathbf{Z}_1'\otimes\mathbf{X}_1)\mathbf{v} = \mathbf{M}_A^{M_B}(\mathbf{I}\otimes\mathbf{\Sigma})\mathbf{g} \wedge (\mathbf{Z}_1\otimes\mathbf{X}_1')\mathbf{g} = \mathbf{p}.$$

We have

$$\mathbf{M}_A^{M_B}(\mathbf{I}\otimes\mathbf{\Sigma})\mathbf{g} + (\mathbf{Z}_1'\otimes\mathbf{X}_1)(\mathbf{Z}_1\otimes\mathbf{X}_1')\mathbf{g} = (\mathbf{Z}_1'\otimes\mathbf{X}_1)(\mathbf{v}+\mathbf{p}),$$
$$\Longrightarrow\ \mathbf{g} = \left[\mathbf{M}_A^{M_B}(\mathbf{I}\otimes\mathbf{\Sigma}) + (\mathbf{Z}_1'\mathbf{Z}_1\otimes\mathbf{X}_1\mathbf{X}_1')\right]^-(\mathbf{Z}_1'\otimes\mathbf{X}_1)(\mathbf{v}+\mathbf{p}).$$

Thus

$$\mathbf{p} = (\mathbf{Z}_1\otimes\mathbf{X}_1')\mathbf{g}$$
$$= (\mathbf{Z}_1\otimes\mathbf{X}_1')\left[\mathbf{M}_A^{M_B}(\mathbf{I}\otimes\mathbf{\Sigma}) + (\mathbf{Z}_1'\mathbf{Z}_1\otimes\mathbf{X}_1\mathbf{X}_1')\right]^-(\mathbf{Z}_1'\otimes\mathbf{X}_1)(\mathbf{v}+\mathbf{p}),$$
$$\Longrightarrow\ \mathbf{v}+\mathbf{p} = \left\{(\mathbf{Z}_1\otimes\mathbf{X}_1')\left[\mathbf{M}_A^{M_B}(\mathbf{I}\otimes\mathbf{\Sigma}) + (\mathbf{Z}_1'\mathbf{Z}_1\otimes\mathbf{X}_1\mathbf{X}_1')\right]^-(\mathbf{Z}_1'\otimes\mathbf{X}_1)\right\}^-\mathbf{p},$$
$$\Longrightarrow\ \mathbf{g} = \left[\mathbf{M}_A^{M_B}(\mathbf{I}\otimes\mathbf{\Sigma}) + (\mathbf{Z}_1'\mathbf{Z}_1\otimes\mathbf{X}_1\mathbf{X}_1')\right]^-(\mathbf{Z}_1'\otimes\mathbf{X}_1)$$
$$\times\left\{(\mathbf{Z}_1\otimes\mathbf{X}_1')\left[\mathbf{M}_A^{M_B}(\mathbf{I}\otimes\mathbf{\Sigma}) + (\mathbf{Z}_1'\mathbf{Z}_1\otimes\mathbf{X}_1\mathbf{X}_1')\right]^-(\mathbf{Z}_1'\otimes\mathbf{X}_1)\right\}^-\mathbf{p}.$$

□

Theorem 3 *The BLUE of the vector function*

$$(\mathbf{Z}_1'\otimes\mathbf{X}_1)vec(\mathbf{B}_1) + (\mathbf{Z}_2'\otimes\mathbf{X}_2)vec(\mathbf{B}_2)$$

under the model (2) is given by

$$[(\mathbf{Z}_1'\otimes\mathbf{X}_1)vec(\widehat{\mathbf{B}_1) + (\mathbf{Z}_2'}\otimes\mathbf{X}_2)vec(\mathbf{B}_2)]_a$$
$$= \left[\mathbf{P}_A^{U^+} + \mathbf{M}_A^{U^+}\mathbf{S}[\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\mathbf{S}]^-\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\right]vec(\mathbf{Y}),$$

where $\mathbf{U} = (\mathbf{I}\otimes\mathbf{\Sigma}) + (\mathbf{Z}_1'\mathbf{Z}_1\otimes\mathbf{X}_1\mathbf{X}_1') + (\mathbf{Z}_2'\mathbf{Z}_2\otimes\mathbf{X}_2\mathbf{X}_2')$*,* $\mathbf{A} = \mathbf{Z}_1'\otimes\mathbf{X}_1$*,* $\mathbf{S} = \mathbf{Z}_2'\otimes\mathbf{X}_2$*.*

Proof According to the Theorem 3.1.3 in [1] we have in the model (2)

$$\widehat{(\mathbf{A},\mathbf{S})\begin{pmatrix} vec(\mathbf{B}_1)\\ vec(\mathbf{B}_2)\end{pmatrix}}_a = (\mathbf{A},\mathbf{S})\left[\begin{pmatrix}\mathbf{A}'\\ \mathbf{S}'\end{pmatrix}^-_{m(I\otimes\Sigma)}\right]' vec(\mathbf{Y})$$

$$= (\mathbf{A},\mathbf{S})\left\{\left[(\mathbf{I}\otimes\mathbf{\Sigma}) + (\mathbf{A},\mathbf{S})\begin{pmatrix}\mathbf{A}'\\ \mathbf{S}'\end{pmatrix}\right]^- (\mathbf{A},\mathbf{S})\left[\begin{pmatrix}\mathbf{A}'\\ \mathbf{S}'\end{pmatrix}\mathbf{U}^-(\mathbf{A},\mathbf{S})\right]^-\right\}' vec(\mathbf{Y}),$$

where $\mathbf{U} = (\mathbf{I}\otimes\mathbf{\Sigma}) + \mathbf{A}\mathbf{A}' + \mathbf{S}\mathbf{S}'$.

Using the following Rohde's formula for generalized inverse of partitioned p.s.d. matrix (see [2], Lemma 13, p. 68)

$$\begin{pmatrix}\mathbf{A}, & \mathbf{B}\\ \mathbf{B}', & \mathbf{C}\end{pmatrix}^- = \begin{pmatrix}\mathbf{A}^- + \mathbf{A}^-\mathbf{B}(\mathbf{C}-\mathbf{B}'\mathbf{A}^-\mathbf{B})^-\mathbf{B}'\mathbf{A}^-, & -\mathbf{A}^-\mathbf{B}(\mathbf{C}-\mathbf{B}'\mathbf{A}^-\mathbf{B})^-\\ -(\mathbf{C}-\mathbf{B}'\mathbf{A}^-\mathbf{B})^-\mathbf{B}'\mathbf{A}^-, & (\mathbf{C}-\mathbf{B}'\mathbf{A}^-\mathbf{B})^-\end{pmatrix}$$

we get

$$\begin{pmatrix}\mathbf{A}'\mathbf{U}^-\mathbf{A}, & \mathbf{A}'\mathbf{U}^-\mathbf{S}\\ \mathbf{S}'\mathbf{U}^-\mathbf{A}, & \mathbf{S}'\mathbf{U}^-\mathbf{S}\end{pmatrix}^- = \begin{pmatrix}\mathbf{A}_{11}, & \mathbf{A}_{12}\\ \mathbf{A}_{21}, & \mathbf{A}_{22}\end{pmatrix},$$

$$\begin{aligned}
\mathbf{A}_{11} &= (\mathbf{A}'\mathbf{U}^-\mathbf{A})^- + (\mathbf{A}'\mathbf{U}^-\mathbf{A})^-\mathbf{A}'\mathbf{U}^-\mathbf{S}\\
&\quad\times[\mathbf{S}'\mathbf{U}^+\mathbf{S} - \mathbf{S}'\mathbf{U}^+\mathbf{A}(\mathbf{A}'\mathbf{U}^-\mathbf{A})^-\mathbf{A}'\mathbf{U}^+\mathbf{S}]^-\mathbf{S}'\mathbf{U}^+\mathbf{A}(\mathbf{A}'\mathbf{U}^-\mathbf{A})^-\\
&= (\mathbf{A}'\mathbf{U}^-\mathbf{A})^- + (\mathbf{A}'\mathbf{U}^-\mathbf{A})^-\mathbf{A}'\mathbf{U}^-\mathbf{S}[\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\mathbf{S}]^-\mathbf{S}'\mathbf{U}^-\mathbf{A}(\mathbf{A}'\mathbf{U}^-\mathbf{A})^-,\\
\mathbf{A}_{12} &= -(\mathbf{A}'\mathbf{U}^-\mathbf{A})^-\mathbf{A}'\mathbf{U}^-\mathbf{S}[\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\mathbf{S}']^- = (\mathbf{A}_{21})',\\
\mathbf{A}_{22} &= [\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\mathbf{S}]^-.
\end{aligned}$$

After some calculations we get

$$\widehat{(\mathbf{A},\mathbf{S})\begin{pmatrix} vec(\mathbf{B}_1)\\ vec(\mathbf{B}_2)\end{pmatrix}}_a = (\mathbf{A},\mathbf{S})$$

$$\times\begin{pmatrix}(\mathbf{A}'\mathbf{U}^-\mathbf{A})^-\mathbf{A}'\mathbf{U}^- - (\mathbf{A}'\mathbf{U}^-\mathbf{A})^-\mathbf{A}'\mathbf{U}^-\mathbf{S}[\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\mathbf{S}]^-\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\\ [\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\mathbf{S}]^-\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\end{pmatrix} vec(\mathbf{Y}).$$

Since $\mathscr{M}(\mathbf{A})\subset\mathscr{M}(\mathbf{U})$, $\mathscr{M}(\mathbf{S})\subset\mathscr{M}(\mathbf{U})$, the expressions $\mathbf{A}'\mathbf{U}^-\mathbf{A}$, $\mathbf{A}'\mathbf{U}^-\mathbf{A}$ are invariant to the choice of g-inverse. Thus using the fact that

$$P\{vec(\mathbf{Y})\in\mathscr{M}[(\mathbf{A},\mathbf{S}),(\mathbf{I}\otimes\mathbf{\Sigma})]\} = 1$$

we can write

$$\widehat{\mathbf{A}vec(\mathbf{B}_1)}_a = \left[\mathbf{P}_A^{U^+} - \mathbf{P}_A^{U^+}\mathbf{S}[\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\mathbf{S}]^-\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\right] vec(\mathbf{Y}),$$

$$\widehat{\mathbf{S}vec(\mathbf{B}_2)}_a = \mathbf{S}[\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\mathbf{S}]^-\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+ vec(\mathbf{Y}),$$

i.e.

$$\widehat{(\mathbf{Z}_1'\otimes\mathbf{X}_1)vec(\mathbf{B}_1)_a + (\mathbf{Z}_2'\otimes\mathbf{X}_2)}vec(\mathbf{B}_2)_a$$

$$= \left[\mathbf{P}_A^{U^+} + \mathbf{M}_A^{U^+}\mathbf{S}[\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\mathbf{S}]^-\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\right] vec(\mathbf{Y}). \quad \square$$

Corollary 1 *Let in the Theorem 3 the condition* $\mathscr{M}(\mathbf{S}) \subset \mathscr{M}(\mathbf{T})$, *where* $\mathbf{T} = (\mathbf{I} \otimes \mathbf{\Sigma}) + \mathbf{A}\mathbf{A}'$, $\mathbf{A} = \mathbf{Z}_1' \otimes \mathbf{X}_1$, $\mathbf{S} = \mathbf{Z}_2' \otimes \mathbf{X}_2$, *is valid. Then*

$$[(\mathbf{Z}_1' \otimes \mathbf{X}_1)vec(\widehat{\mathbf{B}_1) + (\mathbf{Z}_2'} \otimes \mathbf{X}_2)vec(\mathbf{B}_2)]_a$$
$$= \left[\mathbf{P}_A^{T^+} + \mathbf{M}_A^{T^+}\mathbf{S}[\mathbf{S}'(\mathbf{M}_A\mathbf{T}\mathbf{M}_A)^+\mathbf{S}]^-\mathbf{S}'(\mathbf{M}_A\mathbf{T}\mathbf{M}_A)^+\right] vec(\mathbf{Y}).$$

Proof Under the assumption $\mathscr{M}(\mathbf{S}) \subset \mathscr{M}(\mathbf{T})$ one of the matrices

$$\left[\begin{pmatrix} \mathbf{A}' \\ \mathbf{S}' \end{pmatrix}^-_{m(I\otimes\Sigma)}\right]',$$

is the matrix

$$\left[\begin{pmatrix} \mathbf{A}' \\ \mathbf{S}' \end{pmatrix}^-_{m(T)}\right]' = \begin{pmatrix} \mathbf{A}'\mathbf{T}^-\mathbf{A}, & \mathbf{A}'\mathbf{T}^-\mathbf{S} \\ \mathbf{S}'\mathbf{T}^-\mathbf{A}, & \mathbf{S}'\mathbf{T}^-\mathbf{S} \end{pmatrix}^- \begin{pmatrix} \mathbf{A}' \\ \mathbf{S}' \end{pmatrix} \mathbf{T}^-,$$

since
a) this matrix is g-inverse of the matrix $(\mathbf{A}, \mathbf{S})$,
b) the matrix

$$(\mathbf{A}, \mathbf{S}) \begin{pmatrix} \mathbf{A}'\mathbf{T}^-\mathbf{A}, & \mathbf{A}'\mathbf{T}^-\mathbf{S} \\ \mathbf{S}'\mathbf{T}^-\mathbf{A}, & \mathbf{S}'\mathbf{T}^-\mathbf{S} \end{pmatrix}^- \begin{pmatrix} \mathbf{A}'\mathbf{T}^- \\ \mathbf{S}'\mathbf{T}^- \end{pmatrix} (\mathbf{I} \otimes \mathbf{\Sigma})$$

$$= (\mathbf{A}, \mathbf{S}) \left[\begin{pmatrix} \mathbf{A}' \\ \mathbf{S}' \end{pmatrix}^-_{m(T)}\right]' \mathbf{T} - (\mathbf{A}, \mathbf{S}) \begin{pmatrix} \mathbf{A}'\mathbf{T}^-\mathbf{A}, & \mathbf{A}'\mathbf{T}^-\mathbf{S} \\ \mathbf{S}'\mathbf{T}^-\mathbf{A}, & \mathbf{S}'\mathbf{T}^-\mathbf{S} \end{pmatrix}^- \begin{pmatrix} \mathbf{A}'\mathbf{T}^-\mathbf{A} \\ \mathbf{S}'\mathbf{T}^-\mathbf{A} \end{pmatrix} \mathbf{A}'$$

$$= (\mathbf{A}, \mathbf{S}) \left[\begin{pmatrix} \mathbf{A}' \\ \mathbf{S}' \end{pmatrix}^-_{m(T)}\right]' \mathbf{T} - \mathbf{A}\mathbf{A}',$$

is symmetrical. Here the relation [valid under the assumption $\mathscr{M}(\mathbf{S}) \subset \mathscr{M}(\mathbf{T})$]

$$(\mathbf{A}, \mathbf{S}) \begin{pmatrix} \mathbf{A}'\mathbf{T}^-\mathbf{A}, & \mathbf{A}'\mathbf{T}^-\mathbf{S} \\ \mathbf{S}'\mathbf{T}^-\mathbf{A}, & \mathbf{S}'\mathbf{T}^-\mathbf{S} \end{pmatrix}^- \begin{pmatrix} \mathbf{A}'\mathbf{T}^-\mathbf{A}, & \mathbf{A}'\mathbf{T}^-\mathbf{S} \\ \mathbf{S}'\mathbf{T}^-\mathbf{A}, & \mathbf{S}'\mathbf{T}^-\mathbf{S} \end{pmatrix} = (\mathbf{A}, \mathbf{S}),$$

was utilized. Thus enables us to use the matrix $\mathbf{T}$ instead of the matrix $\mathbf{U}$ in the assertion of the Theorem 3. □

Theorem 4 *The variance of the BLUE of the function*

$$\mathbf{g}'\mathbf{M}_{Z_2'\otimes X_2}(\mathbf{Z}_1' \otimes \mathbf{X}_1)vec(\mathbf{B}_1), \quad \mathbf{g} \in R^{mn},$$

in the model (2) is given by

$$
\begin{aligned}
& var[\mathbf{g}'\mathbf{M}_{Z_2'\otimes X_2}\widehat{(\mathbf{Z}_1'\otimes\mathbf{X}_1)vec}(\mathbf{B}_1)]_a = \\
&= var\Big[\mathbf{g}'\mathbf{M}_{Z_2'\otimes X_2}\Big\{\mathbf{P}_A^{U+} - \mathbf{P}_A^{U+}\mathbf{S}[\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\mathbf{S}]^-\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\Big\}\, vec(\mathbf{Y})\Big]_a \\
&= \mathbf{g}'\mathbf{M}_S\Big[\mathbf{A}(\mathbf{A}'\mathbf{U}^-\mathbf{A})^-\mathbf{A}' - \mathbf{A}\mathbf{A}' + \mathbf{A}(\mathbf{A}'\mathbf{U}^-\mathbf{A})^-\mathbf{A}'\mathbf{U}^-\mathbf{S} \\
&\times \big\{[\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\mathbf{S}]^- - \mathbf{I}\big\}\,\mathbf{S}'\mathbf{U}^-\mathbf{A}(\mathbf{A}'\mathbf{U}^-\mathbf{A})^-\mathbf{A}' \\
&+ \mathbf{A}(\mathbf{A}'\mathbf{U}^-\mathbf{A})^-\mathbf{A}'\mathbf{U}^-\mathbf{S}[\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\mathbf{S}][\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\mathbf{S}]^+ \\
&\times \mathbf{S}'\mathbf{U}^-\mathbf{A}(\mathbf{A}'\mathbf{U}^-\mathbf{A})^-\mathbf{A}'\Big]\mathbf{M}_S\mathbf{g}.
\end{aligned}
$$

Proof We get the assertion after some calculations using the facts that

$$
\begin{aligned}
&[\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\mathbf{S}][\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\mathbf{S}][\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\mathbf{S}]^+ \\
&= [\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\mathbf{S}]\mathbf{P}_{[S'(M_AUM_A)^+S]} = [\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\mathbf{S}],
\end{aligned}
$$

$$
\mathbf{U}\mathbf{U}^+\mathbf{A} = \mathbf{A}, \quad (\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\mathbf{A} = \mathbf{O},
$$

and that the expressions are invariant to the choice of g-inverses (since it is the variance of the BLUE). □

Remark 1 For the variances

$$
var[\mathbf{g}'\mathbf{M}_{Z_2'\otimes X_2}\widehat{(\mathbf{Z}_1'\otimes\mathbf{X}_1)vec}(\mathbf{B}_1)], \quad \mathbf{g}\in R^{mn}
$$

in the model (2) and in the model (3) holds

$$
\begin{aligned}
& var[\mathbf{g}'\mathbf{M}_{Z_2'\otimes X_2}\widehat{(\mathbf{Z}_1'\otimes\mathbf{X}_1)}vec(\mathbf{B}_1)] = \mathbf{g}'\mathbf{M}_S[\mathbf{A}(\mathbf{A}'\mathbf{T}^+\mathbf{A})^-\mathbf{A}' - \mathbf{A}\mathbf{A}']\mathbf{M}_S\mathbf{g} \\
&\le var[\mathbf{g}'\mathbf{M}_{Z_2'\otimes X_2}\widehat{(\mathbf{Z}_1'\otimes\mathbf{X}_1)}vec(\mathbf{B}_1)]_a \\
&= \mathbf{g}'\mathbf{M}_S\Big[\mathbf{A}(\mathbf{A}'\mathbf{U}^-\mathbf{A})^-\mathbf{A}' - \mathbf{A}\mathbf{A}' + \mathbf{A}(\mathbf{A}'\mathbf{U}^-\mathbf{A})^-\mathbf{A}'\mathbf{U}^-\mathbf{S} \\
&\times \big\{[\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\mathbf{S}]^- - \mathbf{I}\big\}\,\mathbf{S}'\mathbf{U}^-\mathbf{A}(\mathbf{A}'\mathbf{U}^-\mathbf{A})^-\mathbf{A}' + \mathbf{A}(\mathbf{A}'\mathbf{U}^-\mathbf{A})^-\mathbf{A}'\mathbf{U}^-\mathbf{S} \\
&\times [\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\mathbf{S}][\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\mathbf{S}]^+\mathbf{S}'\mathbf{U}^-\mathbf{A}(\mathbf{A}'\mathbf{U}^-\mathbf{A})^-\mathbf{A}'\Big]\mathbf{M}_S\mathbf{g}.
\end{aligned}
$$

The inequality is a consequence of the fact, that

$$
\mathbf{A}(\mathbf{A}'\mathbf{T}^+\mathbf{A})^-\mathbf{A}' \underset{L}{\le} \mathbf{A}(\mathbf{A}'\mathbf{U}^-\mathbf{A})^-\mathbf{A}'
$$

and that the other two matrices are p.s.d. The matrix

$$\mathbf{A}(\mathbf{A}'\mathbf{U}^-\mathbf{A})^-\mathbf{A}'\mathbf{U}^-\mathbf{S}\left\{[\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\mathbf{S}]^- - \mathbf{I}\right\}\mathbf{S}'\mathbf{U}^-\mathbf{A}(\mathbf{A}'\mathbf{U}^-\mathbf{A})^-\mathbf{A}$$
$$= \mathbf{P}_A^{U^+}\mathbf{S}\{[\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\mathbf{S}]^- - \mathbf{I}\}\mathbf{S}'(\mathbf{P}_A^{U^+})'$$
$$= \mathbf{P}_A^{U^+}\mathbf{S}\{[\mathbf{S}'\mathbf{U}^+\mathbf{S} - \mathbf{S}'\mathbf{U}^+\mathbf{A}(\mathbf{A}'\mathbf{U}^+\mathbf{A})^-\mathbf{A}'\mathbf{U}^+\mathbf{S}]^- - \mathbf{I}\}\mathbf{S}'(\mathbf{P}_A^{U^+})'$$
$$= \mathbf{P}_A^{U^+}\mathbf{S}\{(\mathbf{S}'\mathbf{U}^+\mathbf{S})^- + (\mathbf{S}'\mathbf{U}^+\mathbf{S})^-\mathbf{S}'\mathbf{U}^+\mathbf{A}[\mathbf{A}'(\mathbf{M}_S\mathbf{U}\mathbf{M}_S)^+\mathbf{A}]^+$$
$$\times \mathbf{A}'\mathbf{U}^+\mathbf{S}(\mathbf{S}'\mathbf{U}^+\mathbf{S})^- - \mathbf{I}\}\mathbf{S}'(\mathbf{P}_A^{U^+})'$$
$$= \mathbf{P}_A^{U^+}\mathbf{S}\{(\mathbf{S}'\mathbf{U}^+\mathbf{S})^+ - \mathbf{I}\}\mathbf{S}'(\mathbf{P}_A^{U^+})'$$
$$+ \mathbf{P}_A^{U^+}\mathbf{S}(\mathbf{S}'\mathbf{U}^+\mathbf{S})^-\mathbf{S}'\mathbf{U}^+\mathbf{A}[\mathbf{A}'(\mathbf{M}_S\mathbf{U}\mathbf{M}_S)^+\mathbf{A}]^+\mathbf{A}'\mathbf{U}^+\mathbf{S}(\mathbf{S}'\mathbf{U}^+\mathbf{S})^-\mathbf{S}'(\mathbf{P}_A^{U^+})',$$

is positive semidefinite because $\mathbf{S}[(\mathbf{S}'\mathbf{U}^+\mathbf{S})^+ - \mathbf{I}]\mathbf{S}'$ is p.s.d. It can be proved as follows (see considerations next the Corollary 1.11.6 in [4]):

$$\mathbf{U} = (\mathbf{I}\otimes\boldsymbol{\Sigma}) + \mathbf{A}\mathbf{A}' + \mathbf{S}\mathbf{S}' \overset{\geq}{_L} \mathbf{S}\mathbf{S}' \iff \mathbf{U}^+ \overset{\leq}{_L} (\mathbf{S}\mathbf{S}')^+,$$
$$\Longrightarrow \mathbf{S}'\mathbf{U}^+\mathbf{S} \overset{\leq}{_L} \mathbf{S}'(\mathbf{S}\mathbf{S}')^+\mathbf{S} \iff (\mathbf{S}'\mathbf{U}^+\mathbf{S})^+ \overset{\geq}{_L} [\mathbf{S}'(\mathbf{S}\mathbf{S}')^+\mathbf{S}]^+ = \mathbf{S}'(\mathbf{S}\mathbf{S}')^+\mathbf{S},$$
$$\Longrightarrow \mathbf{S}(\mathbf{S}'\mathbf{U}^+\mathbf{S})^+\mathbf{S}' \overset{\leq}{_L} \mathbf{S}\mathbf{S}'(\mathbf{S}\mathbf{S}')^+\mathbf{S}\mathbf{S}' = \mathbf{S}\mathbf{S}' \iff \mathbf{S}[(\mathbf{S}'\mathbf{U}^+\mathbf{S})^+ - \mathbf{I}]\mathbf{S}' \overset{\geq}{_L} \mathbf{O}.$$

The matrix

$$\mathbf{A}(\mathbf{A}'\mathbf{U}^-\mathbf{A})^-\mathbf{A}'\mathbf{U}^-\mathbf{S}[\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\mathbf{S}][\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\mathbf{S}]^+$$
$$\times \mathbf{S}'\mathbf{U}^-\mathbf{A}(\mathbf{A}'\mathbf{U}^-\mathbf{A})^-\mathbf{A},$$

is also p.s.d. since $[\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\mathbf{S}][\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\mathbf{S}]^+$ is a projection matrix.

We need to find a class of such functions of the useful parameters which are unbiasedly estimable in both models (2), (3) and estimators of which have the same variance. Thus we consider the functions from the class $\mathscr{E}_a$ only.

In [5] was proved (see Theorem 1) that under condition (4) the class of functios mentioned above is

$$\{\mathbf{g}'\mathbf{M}_{Z_2'\otimes X_2}(\mathbf{Z}_1'\otimes\mathbf{X}_1)vec(\mathbf{B}_1):$$
$$(\mathbf{Z}_1\otimes\mathbf{X}_1')\mathbf{M}_{Z_2'\otimes X_2}\mathbf{g}\in\mathscr{M}[(\mathbf{Z}_1\otimes\mathbf{X}_1')(\mathbf{I}\otimes\boldsymbol{\Sigma})(\mathbf{Z}_1'\otimes\mathbf{X}_1)\mathbf{M}_{(Z_1\otimes X_1')(I\otimes\Sigma)(Z_2'\otimes X_2)}\}.$$

From the Remark it is obvious that in the general case it is impossible to find conditions uder which

$$var[\mathbf{g}'\mathbf{M}_{Z_2'\otimes X_2}\widehat{(\mathbf{Z}_1'\otimes\mathbf{X}_1)vec(\mathbf{B}_1)}] = var[\mathbf{g}'\mathbf{M}_{Z_2'\otimes X_2}\widehat{(\mathbf{Z}_1'\otimes\mathbf{X}_1)vec(\mathbf{B}_1)}]_a.$$

If we confine us to the situation when the condition

$$\mathscr{M}(\mathbf{S})\subset\mathscr{M}(\mathbf{T}), \tag{10}$$

i.e.

$$\mathscr{M}(\mathbf{Z}_2'\otimes\mathbf{X}_2)\subset\mathscr{M}[(\mathbf{I}\otimes\boldsymbol{\Sigma}) + (\mathbf{Z}_1'\mathbf{Z}_1\otimes\mathbf{X}_1\mathbf{X}_1')],$$

is valid, it is possible to prove following statement (see [4], Theorem 1.11.7).

Theorem 5 *Let in model (2) the condition (10) be true. Then*

$$var[\mathbf{g}'\mathbf{M}_{Z_2'\otimes X_2}\widehat{(\mathbf{Z}_1'\otimes\mathbf{X}_1)vec(\mathbf{B}_1)}] = var[\mathbf{g}'\mathbf{M}_{Z_2'\otimes X_2}\widehat{(\mathbf{Z}_1'\otimes\mathbf{X}_1)vec(\mathbf{B}_1)}]_a,$$

if and only if

$$(\mathbf{Z}_1\otimes\mathbf{X}_1')\mathbf{M}_{Z_2'\otimes X_2}\mathbf{g}\in\mathscr{M}\left[(\mathbf{Z}_1\otimes\mathbf{X}_1')\mathbf{T}^+(\mathbf{Z}_1'\otimes\mathbf{X}_1)\mathbf{M}_{(Z_1\otimes X_1')T^+(Z_2'\otimes X_2)}\right].$$

Proof Using notation $\mathbf{A}=\mathbf{Z}_1'\otimes\mathbf{X}_1$, $\mathbf{S}=\mathbf{Z}_2'\otimes\mathbf{X}_2$ and condition (10), we have in the model (2)

$$
\begin{gathered}
var[\mathbf{g}'\widehat{\mathbf{M}_S\mathbf{A}vec(\mathbf{B}_1)}_a] = \\
= var[\mathbf{g}'\mathbf{M}_S\{\mathbf{P}_A^{T^+} - \mathbf{P}_A^{T^+}\mathbf{S}[\mathbf{S}'(\mathbf{M}_A\mathbf{T}\mathbf{M}_A)^+\mathbf{S}]^-\mathbf{S}'(\mathbf{M}_A\mathbf{T}\mathbf{M}_A)^+\}vec(\mathbf{Y})] \\
= \mathbf{g}'\mathbf{M}_S\{\mathbf{P}_A^{T^+} - \mathbf{P}_A^{T^+}\mathbf{S}[\mathbf{S}'(\mathbf{M}_A\mathbf{T}\mathbf{M}_A)^+\mathbf{S}]^-\mathbf{S}'(\mathbf{M}_A\mathbf{T}\mathbf{M}_A)^+\}(\mathbf{T}-\mathbf{A}\mathbf{A}') \\
\times\{\mathbf{P}_A^{T^+} - \mathbf{P}_A^{T^+}\mathbf{S}[\mathbf{S}'(\mathbf{M}_A\mathbf{T}\mathbf{M}_A)^+\mathbf{S}]^-\mathbf{S}'(\mathbf{M}_A\mathbf{T}\mathbf{M}_A)^+\}'\mathbf{M}_S\mathbf{g} \\
= \mathbf{g}'\mathbf{M}_S\{\mathbf{P}_A^{T^+}\mathbf{T} - \mathbf{P}_A^{T^+}\mathbf{S}[\mathbf{S}'(\mathbf{M}_A\mathbf{T}\mathbf{M}_A)^+\mathbf{S}]^-\mathbf{S}'(\mathbf{M}_A\mathbf{T}\mathbf{M}_A)^+\mathbf{T} - \mathbf{A}\mathbf{A}'\} \\
\times\{\mathbf{P}_A^{T^+} - \mathbf{P}_A^{T^+}\mathbf{S}[\mathbf{S}'(\mathbf{M}_A\mathbf{T}\mathbf{M}_A)^+\mathbf{S}]^-\mathbf{S}'(\mathbf{M}_A\mathbf{T}\mathbf{M}_A)^+\}'\mathbf{M}_S\mathbf{g} \\
= \mathbf{g}'\mathbf{M}_S\{\mathbf{A}(\mathbf{A}'\mathbf{T}^+\mathbf{A})^-\mathbf{A}' - \mathbf{A}\mathbf{A}' \\
+ \mathbf{A}(\mathbf{A}'\mathbf{T}^+\mathbf{A})^-\mathbf{A}'\mathbf{T}^+\mathbf{S}[\mathbf{S}'(\mathbf{M}_A\mathbf{T}\mathbf{M}_A)^+\mathbf{S}]^+\mathbf{S}'\mathbf{T}^+\mathbf{A}(\mathbf{A}'\mathbf{T}^+\mathbf{A})^-\mathbf{A}'\}\mathbf{M}_S\mathbf{g} \\
= var[\mathbf{g}'\widehat{\mathbf{M}_S\mathbf{A}vec(\mathbf{B}_1)}] \\
+ \mathbf{g}'\mathbf{M}_S\mathbf{A}(\mathbf{A}'\mathbf{T}^+\mathbf{A})^-\mathbf{A}'\mathbf{T}^+\mathbf{S}[\mathbf{S}'(\mathbf{M}_A\mathbf{T}\mathbf{M}_A)^+\mathbf{S}]^+\mathbf{S}'\mathbf{T}^+\mathbf{A}(\mathbf{A}'\mathbf{T}^+\mathbf{A})^-\mathbf{A}'\mathbf{M}_S\mathbf{g}.
\end{gathered}
$$

The second term is zero iff

$$\mathbf{g}'\mathbf{M}_S\mathbf{A}(\mathbf{A}'\mathbf{T}^+\mathbf{A})^-\mathbf{A}'\mathbf{T}^+\mathbf{S}=\mathbf{o}'.$$

It is equivalent to

$$(\mathbf{A}'\mathbf{T}^+\mathbf{A})^-\mathbf{A}'\mathbf{M}_S\mathbf{g}\in\mathscr{M}(\mathbf{M}_{A'T^+S}) \iff \mathbf{A}'\mathbf{M}_S\mathbf{g}\in\mathscr{M}[\mathbf{A}'\mathbf{T}^+\mathbf{A}\mathbf{M}_{A'T^+S}].$$

In the course of the proof the relations $(\mathbf{M}_A\mathbf{T}\mathbf{M}_A)^+\mathbf{A}=\mathbf{O}$, $\mathbf{T}\mathbf{T}^+\mathbf{A}=\mathbf{A}$, $(\mathbf{A}'\mathbf{T}^+\mathbf{A})(\mathbf{A}'\mathbf{T}^+\mathbf{A})^+\mathbf{A}'=\mathbf{A}'$ and the fact, that the expressions are invariant to the choice of the g-inverses have been utilized. □

References

[1] Kubáček, L., Kubáčková, L., Volaufová, J.: Statistical Models with Linear Structures. *Veda, Publishing House of the Slovak Academy of Sciences, Bratislava*, 1995.

[2] Kubáčková, L., Kubáček, L.: *Elimination Transformation of an Observation Vector preserving Information on the First and Second Order Parameters.* Technical Report, No 11, 1990, Institute of Geodesy, University of Stuttgart, 1–71.

[3] Kubáček, L., Kubáčková, L.: Statistika a metrologie. *Vydavatelství UP, Olomouc*, 2000.

[4] Fišerová, E., Kubáček, L., Kunderová, P.: Linear Statistical Models: Regularity and Singularities. *Academia, Praha*, in preparation.

[5] Kunderová, P.: *On one Multivariate linear model with nuisance parameters.* Acta Univ. Palacki. Olomuc., Fac. rer. nat., Math. **36** (1997), 131–139.

[6] Kunderová, P.: *Locally best and uniformly best estimators in linear model with nuisance parameters.* Tatra Mt. Math. Publ. **22** (2001), 27–36.

[7] Kunderová, P.: *On eliminating transformations for nuisance parameters in multivariate linear model.* Acta Univ. Palacki. Olomuc., Fac. rer. nat., Math. **43** (2004), 87–104.

[8] Nordström, K., Fellman, J.: *Characterizations and dispersion-matrix robustness of efficiently estimable parametric functionals in linear models with nuisance parameters.* Linear Algebra and its Applications **127** (1990), 341–361.

Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica **44** (2005) 71–82

Remarks on Existence of Positive Solutions of some Integral Equations

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Abstract

We study the existence of positive solutions of the integral equation

$$x(t) = \mu \int_0^1 k(t,s) f(s, x(s), x'(s), \dots, x^{(n-1)}(s))\, ds, \quad n \geq 2$$

in both $C^{n-1}[0,1]$ and $W^{n-1,p}[0,1]$ spaces, where $p \geq 1$ and $\mu > 0$. Throughout this paper k is nonnegative but the nonlinearity f may take negative values. The Krasnosielski fixed point theorem on cone is used.

Key words: Positive solutions, Fredholm integral equations, cone, boundary value problems, fixed point theorem.

2000 Mathematics Subject Classification: 34G20, 34K10, 34B10, 34B15

4 Introduction

In analyzing nonlinear phenomena many mathematical models give rise to problems for which only nonnegative solutions make sense. This paper deals with existence of positive solutions of the integral equations of the form

$$x(t) = \mu \int_0^1 k(t,s) f(s, x(s), s'(s), \dots, x^{(n-1)}(s))\, ds, \tag{1.1}$$

where $\mu > 0$ is a constant and $n \geq 2$.

Throughout this paper k is nonnegative but our nonlinearity f may take negative values. The literature on positive solutions is for the most part devoted to (1.1), when f takes nonnegative values and f is not dependent on derivatives of the function x (see [2]–[5]). Existence in this paper will be established using Krasnosielskii's fixed point theorem in a cone, which we state here for the convenience of the reader.

Theorem 4.1 (K. Deimling [4], D. Guo [5]). *Let $E = (E, \|\cdot\|)$ be a Banach space and let $K \subset E$ be a cone in E. Assume Ω_1 and Ω_2 are bounded and open subsets of E with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$ and let $A : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be continuous and completely continuous. In addition suppose either $\|Au\| \leq \|u\|$ for $u \in K \cap \partial\Omega_1$ and $\|Au\| \geq \|u\|$ for $u \in K \cap \partial\Omega_2$ or $\|Au\| \geq \|u\|$ for $u \in K \cap \partial\Omega_1$ and $\|Au\| \leq \|u\|$ for $u \in K \cap \partial\Omega_2$ hold. Then A has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.*

5 Main results

In this section we present some results for the integral equation (1.1).

Throughout the paper

$$I = [0,1] \times [0,\infty) \times (-\infty,\infty)^{n-1}, \quad J = [0,\infty) \times (-\infty,\infty)^{n-1}$$

and

$$\|x\|_{n-1} = \sup_{t\in[0,1]} \left[|x(t)| + |x'(t)| + \ldots + |x^{(n-1)}(t)| \right],$$

where $x \in C^{n-1}[0,1]$.

Theorem 5.1 *Suppose the following conditions are satisfied:*

(2.1) *$k : [0,1] \times [0,1] \to [0,\infty)$, $\frac{\partial^l k(t,s)}{\partial t^l}$ ($l = 0,1,\ldots,n-2$) exist and are continuous on $[0,1] \times [0,1]$,*

(2.2) *there exists $\frac{\partial^{n-1} k(t,s)}{\partial t^{n-1}}$ for all $t \in [0,1]$ and a.e. $s \in [0,1]$,*

(2.3) *there exist $k^* \in C[0,1], \overline{k}_i \in L^1[0,1]$ and $M > 0$ such that*

(a) *$k^*(t) > 0$ for a.e. $t \in [0,1]$,*

(b) *$\overline{k}_i(s) \geq 0$ and $\int_0^1 \overline{k}_i(s)ds > 0$ for $i = 0,1,\ldots,n-1$ and a.e. $s \in [0,1]$,*

(c) *$Mk^*(t)\overline{k}_i(s) \leq \left|\frac{\partial^i k(t,s)}{\partial t^i}\right| \leq \overline{k}_i(s)$ for $i = 0,1,\ldots,n-1$; $t \in [0,1]$ and a.e. $s \in [0,1]$,*

(2.4) *the map $t \to \frac{\partial^{n-1}}{\partial t^{n-1}} k(t,s)$ is continuous from $[0,1]$ to $L^1[0,1]$,*

(2.5) *there exists a function* $d \in C[0,1]$ *with* $d(t) > 0$ *for a.e.* $t \in [0,1]$ *such that*

$$k(t,s) - d(t)\left[\left|\frac{\partial k(t,s)}{\partial t}\right| + \ldots + \left|\frac{\partial^{n-1} k(t,s)}{\partial t^{n-1}}\right|\right] \geq d(t)\left[k(t,s) + \left|\frac{\partial k(t,s)}{\partial t}\right| + \ldots + \left|\frac{\partial^{n-1} k(t,s)}{\partial t^{n-1}}\right|\right]$$

for all $t \in [0,1]$ *and a.e.* $s \in [0,1]$,

(2.6) *there exists a constant* $\tilde{c} > 0$ *with*

$$\int_0^1 k(t,s)\,ds \leq \tilde{c} M d(t) k^*(t) \quad \text{for } t \in [0,1],$$

(2.7) $f : I \to (-\infty, \infty)$ *is continuous and there exists a constant* $L > 0$ *with*

$$f(t, v_0, v_1, \ldots, v_{n-1}) + L \geq 0 \quad \text{for } (t, v_0, v_1, \ldots, v_{n-1}) \in I,$$

(2.8) *there exists a function* $\psi(u)$ *such that*

$$f(t, v_0, v_1, \ldots, v_{n-1}) + L \leq \psi(v_0 + |v_1| + \ldots + |v_{n-1}|)$$

on I, *where* $\psi : [0,\infty) \to [0,\infty)$ *is continuous and nondecreasing and* $\psi(u) > 0$ *for* $u > 0$,

(2.9) *there exists* $r > 0$ *such that* $r \geq \mu L \tilde{c}$ *and*

$$\frac{r}{\psi(r + \|\phi\|_{n-1})} \geq \sum_{i=0}^{n-1} \mu \sup_{t \in [0,1]} \int_0^1 \left|\frac{\partial^i k(t,s)}{\partial t^i}\right| ds,$$

where $\phi(t) = \mu L \int_0^1 k(t,s)\,ds$,

(2.10) $f(t, v_0, v_1, \ldots, v_{n-1}) + L \geq g(v_0)$ *for* $(t, v_0, v_1, \ldots, v_{n-1}) \in I$ *with* $g : [0,\infty) \to [0,\infty)$ *continuous and nondecreasing and* $g(u) > 0$ *for* $u > 0$,

(2.11) *there exists* $R > 0$ *and* $t_0 \in [0,1]$ *such that* $R > r$, $k^*(t_0) > 0, d(t_0) > 0$ *and*

$$R \leq \mu \int_0^1 k(t_0,s) + \left[\left|\tfrac{\partial k(t_0,s)}{\partial t}\right| + \ldots + \left|\tfrac{\partial^{n-1} k(t_0,s)}{\partial t^{n-1}}\right|\right] d(t_0) g(\varepsilon R M d(s) k^*(s))\,ds,$$

where $\varepsilon > 0$ *is any constant such that* $1 - \frac{\mu L \tilde{c}}{R} \geq \varepsilon$.

Then (1.1) has a nonnegative solution $x \in C^{n-1}[0,1]$ *with* $x(t) > 0$ *for a.e.* $t \in [0,1]$.

Proof The proof of Theorem 2.1 is similar to that of Theorem 2.1 in the paper [1]. To show (1.1) has a positive solution we will look at

$$x(t) = \mu \int_0^1 k(t,s) f^*(s, x(s) - \phi(s), s'(s) - \phi'(s), \ldots, x^{n-1}(s) - \phi^{(n-1)}(s))\, ds, \tag{2.12}$$

where

$$f^*(t, v_0, v_1, \ldots, v_{n-1}) = \begin{cases} f(t, v_0, v_1, \ldots, v_{n-1}) + L, & \text{if } (t, v_0, v_1, \ldots, v_{n-1}) \in I, \\ f(t, 0, v_1, \ldots, v_{n-1}) + L, & \text{if } (t, v_0, v_1, \ldots, v_{n-1}) \in \tilde{I}, \end{cases}$$

with $\tilde{I} = [0,1] \times (-\infty, 0) \times (-\infty, \infty)^{n-1}$.

We will show that there exists a solution x_1 to (2.12) with $x_1(t) \geq \phi(t)$ for $t \in [0,1]$. If this is true then $u(t) = x_1(t) - \phi(t)$ is a nonnegative solution of (1.1) since for $t \in [0,1]$ we have

$$\begin{aligned} u(t) = & \\ = \mu & \int_0^1 k(t,s) \left[f^*(s, x(s) - \phi(s), x'(s) - \phi'(s), \ldots, x^{(n-1)}(s) - \phi^{(n-1)}(s)) \right] ds \\ & - \mu L \int_0^1 k(t,s)\, ds \\ = \mu & \int_0^1 k(t,s) f(s, x_1(s) - \phi(s), x_1'(s) - \phi'(s), \ldots, x_1^{(n-1)}(s) - \phi^{(n-1)}(s))\, ds \\ = \mu & \int_0^1 k(t,s) f(s, u(s), u'(s), \ldots, u^{(n-1)}(s))\, ds. \end{aligned}$$

We will concentrate our study on (2.12).

Let $E = (C^{(n-1)}[0,1], \|\cdot\|_{n-1})$ and

$$K = \{u \in C^{n-1}[0,1] : u(t) - d(t)\left[|u'(t)| + \ldots + |u^{(n-1)}(t)|\right] \geq M d(t) k^*(t) \|u\|_{n-1}.$$

Clearly K is cone of E. Let

$$\begin{aligned} \Omega_1 &= \{u \in C^{n-1}[0,1] :\ \|u\|_{n-1} < r\}, \\ \Omega_2 &= \{u \in C^{n-1}[0,1] :\ \|u\|_{n-1} < R\} \end{aligned}$$

and

$$\tilde{f}(s, x(s) - \phi(s)) = f^*(s, x(s) - \phi(s), x'(s) - \phi'(s), \ldots, x^{(n-1)} - \phi^{(n-1)}(s)),$$

where $x \in C^{n-1}[0,1]$. Now, let

$$A : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to C^{n-1}[0,1]$$

be defined by

$$(Ax)(t) = \mu \int_0^1 k(t,s) \tilde{f}(s, x(s) - \phi(s))\, ds.$$

First we show $A : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$. If $x \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$ and $t \in [0,1]$, then relations (2.1), (2.5) imply

$$\begin{aligned} &Ax(t) - d(t)[|(Ax)'(t)| + \ldots + |(Ax)^{(n-1)}(t)|] \\ &\geq \mu \int_0^1 k(t,s)\tilde{f}(s, x(s) - \phi(s))\, ds \\ &- \mu\, d(t) \int_0^1 \left[\left| \tfrac{\partial k(t,s)}{\partial t} \right| + \ldots + \left| \tfrac{\partial^{n-1} k(t,s)}{\partial t^{n-1}} \right| \right] \tilde{f}(s, x(s) - \phi(s))\, ds \\ &\geq \mu d(t) \int_0^1 \left[k(t,s) + \left| \tfrac{\partial k}{\partial t}(t,s) \right| + \ldots + \left| \tfrac{\partial^{n-1} k(t,s)}{\partial t^{n-1}} \right| \right] \tilde{f}(s, x(s) - \phi(s))\, ds \end{aligned}$$

and this together with (2.3) yields

$$\begin{aligned} \|Ax\|_{n-1} &\geq Ax(t) - d(t) \left[|(Ax)'(t)| + \ldots + |(Ax)^{(n-1)}(t)| \right] \\ &\geq \mu d(t) \left(\sum_{i=0}^{n-1} M k^*(t) \int_0^1 \overline{k}_i(s) \tilde{f}(s, x(s) - \phi(s))\, ds \right). \end{aligned} \tag{2.13}$$

On the other hand (2.3) implies

$$\|Ax\|_{n-1} \leq \sum_{i=0}^{n-1} \mu \int_0^1 \overline{k}_i(s) \tilde{f}(s, x(s) - \phi(s))\, ds. \tag{2.14}$$

Taking into account (2.13)–(2.14) we conclude that

$$Ax(t) - d(t)[|(Ax)'(t)| + \ldots + |(Ax)^{(n-1)}(t)|] \geq M d(t) k^*(t) \|Ax\|_{n-1} \quad \text{for } t \in [0,1].$$

Consequently $Ax \in K$ so $A : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$. We now show

$$\|Ax\|_{n-1} \leq \|x\|_{n-1} \quad \text{for } x \in K \cap \partial\Omega_1. \tag{2.15}$$

To see this let $x \in K \cap \partial\Omega_1$. Then $\|x\|_{n-1} = r$ and $x(t) \geq M d(t) k^*(t) r$ for $t \in [0,1]$. For $t \in [0,1]$ we have

$$\sum_{i=0}^{n-1} |(Ax)^{(i)}(t)| \leq \sum_{i=0}^{n-1} \int_0^1 \left| \frac{\partial^i k(t,s)}{\partial t^i} \right| \tilde{f}(s, x(s) - \phi(s))\, ds.$$

This together with (2.8)–(2.9) yields

$$\begin{aligned} \|Ax\|_{n-1} &\leq \mu\psi\left(\|x\|_{n-1} + \|\phi\|_{n-1}\right) \sum_{i=0}^{n-1} \sup_{t\in[0,1]} \int_0^1 \left| \frac{\partial^i k(t,s)}{\partial t^i} \right| ds \\ &\leq \mu\psi(r + \|\phi\|_{n-1}) \sum_{i=0}^{n-1} \sup_{t\in[0,1]} \int_0^1 \left| \frac{\partial^i k(t,s)}{\partial t^i} \right| ds \leq r = \|x\|_{n-1}. \end{aligned}$$

So (2.15) holds. Next we show

$$\|Ax\|_{n-1} \geq \|x\|_{n-1} \quad \text{for } x \in K \cap \partial\Omega_2. \tag{2.16}$$

To see it let $x \in K \cap \partial\Omega_2$. Then we get $\|x\|_{n-1} = R$ and $x(t) \geq RMd(t)k^*(t)$ for $t \in [0,1]$. Let ε be as in (2.11). For $t \in [0,1]$ we have from (2.6) that

$$\begin{aligned} x(t) - \phi(t) = x(t) - \mu L \int_0^1 k(t,s)\, ds \geq x(t) - \frac{\mu L \tilde{c} M d(t) k^*(t) R}{R} \\ \geq x(t)\left(1 - \frac{\mu L \tilde{c}}{R}\right) \geq x(t)\varepsilon \geq \varepsilon RMd(t)k^*(t) > 0 \end{aligned}$$

for a.e. $t \in [0,1]$. By (2.10)–(2.11) and (2.5) we have

$$\begin{aligned} \|Ax\|_{n-1} \geq Ax(t_0) - d(t_0)[|(Ax)'(t_0)| + \ldots + |(Ax)^{(n-1)}(t_0)|] \\ \geq \mu d(t_0) \int_0^1 \left[k(t_0,s) + \left|\frac{\partial k(t_0,s)}{\partial t}\right| + \ldots + \left|\frac{\partial^{n-1} k(t_0,s)}{\partial t^{n-1}}\right|\right] g(\varepsilon RMd(s)k^*(s))\, ds \\ \geq R = \|x\|_{n-1}. \end{aligned}$$

Hence we obtain (2.14). By (2.3)–(2.4) and the Arzela–Ascoli theorem we conclude that $A : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ is continuous and compact. Theorem 1.1 implies A has a fixed point $x_1 \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$, i.e. $r \leq \|x_1\|_{n-1} \leq R$ and

$$x_1(t) \geq Md(t)k^*(t)r \quad \text{for } t \in [0,1]. \tag{2.18}$$

Taking into account relations (2.6), (2.9) and (2.18) we have

$$x_1(t) \geq Md(t)k^*(t)r \geq \mu L \tilde{c} M d(t) k^*(t) \geq \mu L \int_0^1 k(t,s)\, ds = \phi(t).$$

This completes the proof of Theorem 2.1. □

Example 5.1 To illustrate the applicability of Theorem 2.1 we consider the following boundary value problem

$$x''(t) + \mu((x(t) + |x'(t)|)^2 - 1) = 0, \quad x(0) = x'(0), \quad x(1) = -x'(1). \tag{2.19}$$

The problem (2.19) is equivalent to the problem of determinig the fixed point of the operator T of the form

$$T(x)(t) = \mu \int_0^1 k(t,s)[(x(s) + |x'(s)|)^2 - 1]\, ds,$$

where $k(t,s)$ is defined as follows

$$k(t,s) = \begin{cases} \frac{(2-t)(1+s)}{3}, & 0 \leq s \leq t \leq 1 \\ \frac{(2-s)(1+t)}{3}, & 0 \leq t \leq s \leq 1. \end{cases}$$

Fix $t_0 = \frac{1}{2}$, $d(t) = M = \frac{1}{4}$, $k^*(t) = 1$, $\overline{k_0}(s) = \overline{k_1}(s) = \frac{4}{3}$, $L = 1$ and $\psi(u) = g(u) = u^2$ for $t \in [0,1]$ and $u \in [0,\infty)$. We claim (2.6) holds with $\tilde{c} = 10$, $\mu < \frac{1}{10}, R > 1$ and $\varepsilon = 1 - \frac{\mu L \tilde{c}}{R} = 1 - \frac{10\mu}{R}$. To see this notice for $t \in [0,1]$ that

$$\int_0^1 k(t,s)\,ds = \frac{1}{2}(1+t-t^2) \leq \frac{5}{8} \leq \tilde{c} M d(t) k^*(t) \leq \frac{\tilde{c}}{16}.$$

Clearly $g(\varepsilon RMd(s)k^*(s)) = \varepsilon^2 R^2 M^2 d^2(s) k^{*2}(s) = \frac{\varepsilon^2 R^2}{256}$ and

$$\begin{aligned}
&\mu d\left(\frac{1}{2}\right)\int_0^1 \left[k\left(\frac{1}{2},s\right) + \left|\frac{\partial k\left(\frac{1}{2},s\right)}{\partial t}\right|\right] g(\varepsilon RMd(s)k^*(s))\,ds \\
&\qquad = \frac{\mu\varepsilon^2 R^2}{1024}\int_0^1 \left[k\left(\frac{1}{2},s\right) + \left|\frac{\partial k\left(\frac{1}{2},s\right)}{\partial t}\right|\right] ds \geq R
\end{aligned}$$

for sufficiently large R. Next we claim (2.9) holds. To see this notice for $t \in [0,1]$ that

$$\phi(t) = \mu L \int_0^1 k(t,s)ds = \frac{\mu}{2}(1+t-t^2)$$

and

$$\|\phi\|_1 = \frac{\mu}{2}\|1-t-t^2\|_1 = \frac{\mu}{2}\sup_{t\in[0,1]}[(1+t-t^2) + |1-2t|] = \mu$$

and

$$\mu\left[\sup_{t\in[0,1]}\int_0^1 k(t,s)ds + \sup_{t\in[0,1]}\int_0^1 \left|\frac{\partial k(t,s)}{\partial t}\right| ds\right] = \frac{9\mu}{8}.$$

Finally notice (2.9) is satisfied with $r = 10\mu$ since $\frac{9}{8}\mu \leq \frac{r}{\psi(r+\mu)} = \frac{10}{121\mu}$ for $\mu \leq \frac{\sqrt{80}}{33}$. Thus all assumptions of Theorem 2.1 are satisfied so existence of a positive solution of the problem (2.19) is guaranted.

It is possible to obtain another existence results for (1.1) if we change some conditions on the nonlinearity f and some of conditions on the kernel k. Before formulating a next theorem we will introduce some notation.

For $p \geq 1, L^p[0,1]$ is the Banach space of all real functions x such that $|x|^p$ is Lebesgue integrable on $[0,1]$ with the norm

$$\|x\|_p^* = \left(\int_0^1 |x(t)|^p\right)^{\frac{1}{p}}.$$

The symbol $W^{n-1,p}[0,1]$ $(n \geq 2)$ denotes the set of all functions x with $x^{(n-2)}$ absolutely continuous and $x^{(n-1)} \in L^p[0,1]$.

For $x \in W^{n-1,p}[0,1]$ we introduce the following norm

$$\|x\|_{n-1,p} = \sup_{t\in[0,1]}\left[\sum_{j=0}^{n-2}|x^{(j)}(t)|\right] + \|x^{(n-1)}\|_p^*.$$

The space $(W^{n-1,p}[0,1], \|\cdot\|_{n-1,p})$ is the Banach space.

We adopt the following convention $y(t+\tau) = 0$ if $t+\tau \notin [0,1]$ and $y \in L^p[0,1]$.
A function $f : I \to (-\infty, \infty)$ is a Carathéodory function provided:
If $f = f(t,z)$, then
(i) the map $z \to f(t,z)$ is continuous for almost all $t \in [0,1]$,
(ii) the map $t \to f(t,z)$ is measurable for all $z \in [0,\infty) \times (-\infty,\infty)^{n-1}$.
If f is a Carathéodory function, by a solution to (1.1) we will mean a function x which has an absolutely continuous $(n-2)$ st derivative such that x satisfies the integral equation (1.1) almost everywhere in $[0,1]$.

Theorem 5.2 *Assume that conditions (2.1)–(2.2) and (2.5) are satisfied and p, q are such that $p, q \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Suppose the following conditions are satisfied*

(2.20) there exist $k^* \in C[0,1], \overline{k}_i \in L^p[0,1]$, $\tilde{c} > 0$ and $M > 0$ such that

(a) $k^*(t) > 0$ for a.e. $t \in [0,1]$,

(b) $\overline{k}_i(s) \geq 0$ and $\int_0^1 \overline{k}_i(s)ds > 0$ for $i = 0, 1, \ldots, n-1$ and a.e. $s \in [0,1]$,

(c) $Mk^*(t)\overline{k}_i(s) \leq \left|\frac{\partial^i k(t,s)}{\partial t^i}\right| \leq \overline{k}_i(s)$ for $i = 0, 1, \ldots, n-1$, $t \in [0,1]$ and a.e. $s \in [0,1]$,

(d) the map $(t,s) \to \frac{\partial^{n-1} k(t,s)}{\partial t^{n-1}}$ is measurable,

(e) $\int_0^1 k(t,s)\, ds \leq \tilde{c} M d(t) k^*(t)$ for $t \in [0,1]$.

(2.21) $f : I \to (-\infty, \infty)$ is a Carathéodory function and there exist nonnegative functions $p_j \in L^q[0,1]$ $(j = 0, 1, \ldots, n-1)$ and constants $L > 0$ and $p_n > 0$ such that

(a) $f(t, v_0, v_1, \ldots, v_{n-1}) + L \geq 0$ for a.e. $t \in [0,1]$ and all $(v_0, v_1, \ldots, v_{n-1}) \in J$,

(b) $|f(t, v_0, v_1, \ldots, v_{n-1})| \leq \sum_{i=0}^{n-2} p_i(t)|v_i| + p_{n-1}(t) + p_n|v_{n-1}|^{\frac{p}{q}}$ for a.e. $t \in [0,1]$ and all $(v_0, v_1, \ldots, v_{n-1}) \in J$,

(c) $f(t, v_0, v_1, \ldots, v_{n-1}) + L \leq \psi(v_0 + |v_1| + \ldots + |v_{n-1}|)$ for a.e. $t \in [0,1]$ and all $(v_0, v_1, \ldots, v_{n-1}) \in J$, where $\psi : [0,\infty) \to [0,\infty)$ is a continuous and nondecreasing with $\psi(u) > 0$ for $u > 0$,

(2.22) $\|\psi(x + |x'| + \ldots + |x^{(n-1)}|)\|_q^* \leq \varphi(\|x\|_{n-1,p})$ with $\varphi : [0,\infty) \to [0,\infty)$ continuous and nondecreasing and $x \in W^{n-1,p}[0,1]$,

(2.23) $f(t, v_0, v_1, \ldots, v_{n-1}) + L \geq g(v_0)$ for a.e. $t \in [0,\infty)$ and all $(v_0, v_1, \ldots, v_{n-1}) \in J$ with $g : [0,\infty) \to [0,\infty)$ continuous and nondecreasing and $g(u) > 0$ for $u > 0$,

(2.24) there exists $r > 0$ such that $r \geq \mu L\tilde{c}$ and

$$\frac{r}{\varphi(r + \|\phi\|_{n-1,p})} \geq \mu(b + \|\overline{k}_{n-1}\|_p^*),$$

where

$$b = \sum_{i=0}^{n-2} \sup_{t\in[0,1]} \left\| \frac{\partial^i k(t,\cdot)}{\partial t^i} \right\|_p^*$$

and ϕ is defined by (2.9),

(2.25) there exist $R > 0$ and $t_0 \in [0,1]$ such that $R > r, k^*(t_0) > 0$, $d(t_0) > 0$ and

$$R \leq \mu \int_0^1 \left[k(t_0,s) + \left|\tfrac{\partial k(t_0,s)}{\partial t}\right| + \ldots + \left|\tfrac{\partial^{n-1} k(t_0,s)}{\partial t^{n-1}}\right| \right] d(t_0) g(\varepsilon RMd(s)k^*(s))ds,$$

where ε is defined by (2.11).

Then (1.1) has a nonnegative solution $x \in W^{n-1,p}[0,1]$ with $x(t) > 0$ for a.e. $t \in [0,1]$.

Proof It is enough to show (2.12) has a solution $u \in W^{n-1,p}[0,1]$. Let $a(t) = Md(t)k^*(t)$ and let

$$K = \{u \in W^{n-1,p}[0,1] : u(t) - d(t)\left[|u'(t)| + \ldots + |u^{(n-1)}(t)|\right] \geq a(t)\|u\|_{n-1,p} \quad \text{for a.e. } t \in [0,1]\}.$$

Clearly K is a cone of $W^{n-1,p}[0,1]$.

Let

$$\Omega_1 = \{x \in W^{n-1,p}[0,1] : \|x\|_{n-1,p} < r\},$$
$$\Omega_2 = \{x \in W^{n-1,p}[0,1] : \|x\|_{n-1,p} < R\}$$

and

$$\tilde{f}(s, x(s) - \phi(s)) = f^*(s, x(s) - \phi(s), x'(s) - \phi'(s), \ldots, x^{(n-1)}(s) - \phi^{(n-1)}(s)),$$

where $x \in W^{n-1,p}[0,1]$ and f^* is defined by (2.12). We will show that there exist a solution $x_1 \in W^{n-1,p}[0,1]$ to the equation (2.12) with $x_1(t) \geq \phi(t)$ for $t \in [0,1]$.

Let $A : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to W^{n-1,p}[0,1]$ be defined by

$$Ax(t) = \mu \int_0^1 k(t,s)\tilde{f}(s, x(s) - \phi(s))\, ds.$$

Then

$$|(Ax)^{(n-1)}(t)| \leq \mu \int_0^1 \overline{k}_n(s)\tilde{f}(s, x(s) - \phi(s))\, ds \tag{2.27}$$

and

$$|Ax(t)| + |(Ax)'(t)| + \ldots + |(Ax)^{(n-2)}(t)| \le \mu \sum_{i=0}^{n-2} \int_0^1 \overline{k}_i(s)\tilde{f}(s, x(s) - \phi(s))\, ds. \tag{2.28}$$

From relations (2.27)–(2.28), (2.21)–(2.22) and Hölder's inequality it follows

$$\begin{aligned}\|Ax\|_{n-1,p} &\le \mu \sum_{i=0}^{n-1} \int_0^1 \overline{k}_i(s)\tilde{f}(s, x(s) - \phi(s))\, ds \\ &\le \mu \sum_{i=0}^{n-1} \varphi(\|x\|_{n-1,p} + \|\phi\|_{n-1,p})\|k_i\|_p^*. \end{aligned} \tag{2.29}$$

Note that A is well defined operator. Now we will prove

$$A : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K.$$

If $x \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$ and $t \in [0,1]$, then (2.20), (2.5) and (2.29) imply

$$\begin{aligned} &Ax(t) - d(t)\left[|(Ax)'(t)| + \ldots + |(Ax)^{(n-1)}(t)|\right] \\ &\ge \mu d(t) \int_0^1 \left[k(t,s) + \left|\frac{\partial k(t,s)}{\partial t}\right| + \ldots + \left|\frac{\partial^{n-1} k(t,s)}{\partial t^{n-1}}\right|\right] \tilde{f}(s, x(s) - \phi(s))\, ds \\ &\ge \mu d(t) M k^*(t) \left(\sum_{i=0}^{n-1} \int_0^1 \overline{k}_i(s)\tilde{f}(s, x(s) - \phi(s))\right) ds \ge a(t)\|Ax\|_{n-1,p}. \end{aligned}$$

Thus $Ax \in K$ and $A : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$. Now we will prove that A is a continuous operator. It is enough to show that the Niemytzki operator $H : W^{n-1,p}[0,1] \to L^q[0,1]$ defined by

$$Hx(t) = f^*(t, x(t) - \phi(t), x'(t) - \phi'(t), \ldots, x^{(n-1)}(t) - \phi^{(n-1)}(t))$$

is continuous. The proof of the continuity of H is similar to the proof of Theorem 1.2 in [6]. Let $\{\overline{x}_\nu\}$ be a sequence of elements of $W^{n-1,p}[0,1]$ converging to $\overline{x}$ in $W^{n-1,p}[0,1]$. Then there exists a subsequence $\{x_{\nu_\lambda}^{(n-1)}(t)\}$ of the sequence $\{x_\nu^{(n-1)}(t)\}$ such that

$$\lim_{\lambda\to\infty} \overline{x}_{\nu_\lambda}^{(n-1)}(t) = \overline{x}^{(n-1)}(t) \quad \text{for a.e. } t \in [0,1].$$

Moreover, there exists a function $g \in L^p[0,1]$ with

$$|\overline{x}_{\nu_\lambda}^{(n-1)}(t)| \le g(t) \quad \text{for a.e. } t \in [0,1]$$

([6], Lemma 2.1). Hence by (2.21)(b) we conclude that there exists a function $h \in L^q[0,1]$ such that

$$\begin{aligned} &|f^*(t, \overline{x}(t) - \phi(t), \overline{x}'(t) - \phi'(t), \ldots, \overline{x}^{(n-1)}(t) - \phi^{(n-1)}(t) \\ &\quad - f^*(t, \overline{x}_{\nu_\lambda}(t) - \phi(t), \overline{x}'_{\nu_\lambda}(t) - \phi'(t), \ldots, \\ &\quad \overline{x}_{\nu_\lambda}^{(n-1)}(t) - \phi^{(n-1)}(t))| \le h(t) \quad \text{for a.e. } t \in [0,1]. \end{aligned}$$

From the Lebesgue dominated convergence theorem it folows that the Niemytzki operator H is continuous at the point $\overline{x}$. We next show that A is completely continuous. Let Ω be a bounded set in $\left(W^{n-1,p}[0,1], \|\cdot\|_{n-1,p}\right)$. Then by virtue of (2.29) we have $A(\Omega)$ is bounded. We need to prove that $A(\Omega)$ is relatively compact. We will use the Arzela–Ascoli and the Riesz theorems. In fact, let $y_\nu \in A(\Omega)$ i.e.

$$y_\nu = A(x_\nu), \quad x_\nu \in \Omega.$$

Since $A(\Omega)$ is bounded in $(W^{n-1,p}[0,1], \|\cdot\|_{n-1,p})$ there exist subsequences $\{x_{\nu_\mu}^{(j)}\}$ and $\{y_{\nu_\mu}^{(j)}\}$ of sequences $\{x_\nu^{(j)}\}$ and $\{y_\nu^{(j)}\}$ uniformly convergent to $x^{(j)}$ and $y^{(j)}$ respectively for $j = 0, 1, \ldots, n-2$. Without loss of generality we can assume that the sequences $\{x_\nu^{(j)}\}$ and $\{y_\nu^{(j)}\}$ are uniformly convergent to $x^{(j)}$ and $y^{(j)}$. We will prove that there exists a subsequence $\{y_{\nu_\lambda}^{(n-1)}\}$ of the sequence $\{y_\nu^{(n-1)}\}$ such that

$$\lim_{\lambda\to\infty} \|y_{\nu_\lambda}^{(n-1)} - \overline{y}\|_p^* = 0, \quad \text{where } \overline{y} \in L^p[0,1].$$

Indeed, for fixed $\tau > 0$ we have by the Hölder inequality and the Fubini theorem that

$$\begin{aligned}
&\int_0^1 \left|(Ax)^{(n-1)}(t+\tau) - (Ax)^{(n-1)}(t)\right|^p dt \le \\
&\le \mu^p \int_0^1 \left(\int_0^1 \left|\frac{\partial^{n-1}}{\partial t^{n-1}}k(t+\tau,s) - \frac{\partial^{n-1}}{\partial t^{n-1}}k(t,s)\right|^p ds\right) dt \\
&\times \int_0^1 \left(\int_0^1 |\tilde{f}(s, x(s) - \phi(s))|^q ds\right)^{\frac{p}{q}} dt \\
&\le \mu^p \varphi\left(\|x\|_{n-1,p} + \|\phi\|_{n-1,p}\right)^p \int_0^1 \left(\int_0^1 \left|\frac{\partial^{n-1}}{\partial t^{n-1}}k(t+\tau,s) - \frac{\partial^{n-1}}{\partial t^{n-1}}k(t,s)\right|^p dt\right) ds.
\end{aligned}$$

Now using the fact that translates of L^p are functions continuous in the norm we see that

$$\int_0^1 \left|(Ax)^{(n-1)}(t+\tau) - (Ax)^{(n-1)}(t)\right|^p dt \to 0$$

as $\tau \to 0$ uniformly. From the Riesz compactness theorem it folows that there exists a subsequence $\{y_{\nu_\lambda}^{(n-1)}\}$ of the sequence $\{y_\nu^{(n-1)}\}$ convergent in $L^p[0,1]$ to a function $\overline{y} \in L^p[0,1]$. It is easy to notice that

$$y^{(n-1)}(t) = \overline{y}(t) \quad \text{for a.e. } t \in [0,1].$$

So $A(\Omega)$ is relatively compact, i.e. A is completely continuous. Next we show that

$$\|Ax\|_{n-1,p} \le \|x\|_{n-1,p} \quad \text{for } x \in K \cap \partial\Omega_1. \tag{2.30}$$

Let $x \in K \cap \partial\Omega_1$, so $\|x\|_{n-1,p} = r$ and $x(t) \ge a(t)r$ for a.e. $t \in [0,1]$. The relations (2.21)–(2.22), (2.24), (2.27)–(2.29) yield

$$\sum_{j=0}^{n-2} \left|(Ax)^{(j)}(t)\right| \le \mu b \varphi(\|x\|_{n-1,p} + \|\phi\|_{n-1,p}) \tag{2.31}$$

and

$$\sum_{j=0}^{n-2} |(Ax)^{(j)}(t)| + \|Ax\|_p^* \leq \mu\varphi(\|x\|_{n-1,p} + \|\phi\|_{n-1,p})(b + \|\overline{k}_{n-1}\|_p^*) \leq r \quad (2.32)$$

By (2.31)–(2.32) and (2.24) we get

$$\|Ax\|_{n-1,p} \leq \|x\|_{n-1,p}.$$

So (2.30) holds. Using arguments similar to these in the proof of Theorem 2.1 we conclude that

$$\|Ax\|_{n-1,p} \geq \|x\|_{n-1,p} \quad \text{for } x \in K \cap \partial\Omega_2.$$

Theorem 1.1 implies A has a fixed point $x_1 \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$ i.e.

$$r \leq \|x_1\|_{n-1,p} \leq R \quad \text{and} \quad x_1(t) \geq a(t)r.$$

Thus for a.e. $t \in [0,1]$ we have $x_1(t) \geq a(t)r \geq \phi(t)$. This completes the proof of Theorem 2.3. □

References

[1] Agarwal, R. P., Grace, S. R., O'Regan, D.: *Existence of positive solutions of semipositone Fredholm integral equation.* Funkcialaj Equaciaj **45** (2002), 223–235.

[2] Agarwal, R. P., O'Regan, D.: Infinite Interval Problems For Differential, Difference and Integral Equations. *Kluwer Acad. Publishers, Dordrecht, Boston, London*, 2001.

[3] Agarwal, R. P., O'Regan, D., Wang, J. Y.: Positive Solutions of Differential, Difference and Integral Equations. *Kluwer Academic Publishers, Dordrecht, Boston, London*, 1999.

[4] Deimling, K.: Nonlinear Functional Analysis. *Springer, New York*, 1985.

[5] Guo, D., Lakshmikannthan, V.: Nonlinear Problems in Abstract Cones. *Academic Press, San Diego*, 1988.

[6] Galewski, A.: *On a certain generalization of the Krasnosielskii theorem.* J. Appl. Anal. **1** (2003), 139–147.

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Mathematica **44** (2005) 83–96

Weak and Strong Convergence Theorems of Common Fixed Points for a Pair of Nonexpansive and Asymptotically Nonexpansive Mappings *

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Abstract

The purpose of this paper is to establish some weak and strong convergence theorems of modified three-step iteration methods with errors with respect to a pair of nonexpansive and asymptotically nonexpansive mappings in uniformly convex Banach spaces. The results presented in this paper generalize, improve and unify a few results due to Chang [1], Liu and Kang [5], Osilike and Aniagbosor [7], Rhoades [8] and Schu [9], [10] and others. An example is included to demonstrate that our results are sharp.

Key words: Nonexpansive mappings, asymptotically nonexpansive mappings, common fixed points, modified three-step iteration methods with errors with respect to a pair of mappings.

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1 Introduction

In 1972, Goebel and Kirk [3] introduced the concept of asymptotically nonexpansive mappings and proved that if K is a nonempty closed bounded subset of a uniformly convex Banach space E, then every asymptotically nonexpansive self-mapping of K has a fixed point. After that, some authors studied a few iterative approximation methods of fixed points for asymptotically nonexpansive mappings. In 1991, Schu [9], [10] introduced the modified Ishikawa iteration methods and modified Mann iteration methods and proved that the modified Mann iteration sequence converges strongly to some fixed points of asymptotically nonexpansive mappings in Hilbert spaces. Rhoades [8] extended the results in [9] to uniformly convex Banach spaces and to modified Ishikawa iteration methods. Chang [1], Liu and Kang [5] and Osilike and Aniagbosor [7] also established some strong and weak convergence theorems of modified Ishikawa iteration methods with errors and three-step iteration methods with errors for asymptotically nonexpansive mappings.

Inspired and motivated by the work in [1], [5] and [7]–[10], in this paper we introduce a new iterative method, called modified three-step iteration method with errors with respect to a pair of mappings, and establish some strong and weak convergence theorems of the modified three-step iteration method with errors with respect to nonexpansive and asymptotically nonexpansive mappings in nonempty closed convex subsets of uniformly convex Banach spaces. The results presented in this paper generalize, improve and unify a few results due to Chang [1], Liu and Kang [5], Osilike and Aniagbosor [7], Rhoades [8] and Schu [9], [10] and others. An example is included to demonstrate that our results are sharp.

2 Preliminaries

Let E be a uniformly convex Banach space, K be a nonempty subset of E and $S, T : K \to K$ be two mappings. I stands for the identity mapping, $F(T)$ and $F(S,T)$ denote the sets of fixed points of T and common fixed points of S and T, respectively. Let $J : E \to 2^E$ be the normalized duality mapping defined by

$$J(x) = \{f \in E^*, \langle x, f\rangle = \|x\| \cdot \|f\|, \|f\| = \|x\|\}, \quad \forall x \in E.$$

Let us recall the following concepts and results.

Definition 2.1 [2] A mapping $T : K \to K$ is said to be

(1) *asymptotically nonexpansive* if there exists a sequence $\{k_n\}_{n\geq 1} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that $\|T^n x - T^n y\| \leq k_n\|x - y\|$, $\forall x, y \in K$, $n \geq 1$;

(2) *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$, $\forall x, y \in K$;

(3) *uniformly L-Lipschitzian* if there exists a constant $L \geq 1$ satisfying $\|T^n x - T^n y\| \leq L\|x - y\|$, $\forall x, y \in K$, $n \geq 1$;

(4) *semi-compact* if K is closed and for any bounded sequence $\{x_n\}_{n\geq 1}$ in K with $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$, there exists a subsequence $\{x_{n_i}\}_{i\geq 1} \subset \{x_n\}_{n\geq 1}$ and $x \in K$ such that $\lim_{i\to\infty} x_{n_i} = x$.

It is easy to see that if T is an asymptotically nonexpansive mapping with a sequence $\{k_n\}_{n\geq 1} \subset [1,\infty)$ such that $\lim_{n\to\infty} k_n = 1$, then it must be uniformly L-Lipschitzian with $L = \sup\{k_n : n \geq 1\}$.

Definition 2.2 A mapping T with domain $D(T)$ and range $R(T)$ in E is called *demiclosed* at a point $p \in D(T)$ if whenever $\{x_n\}_{n\geq 1}$ is a sequence in E which converges weakly to a point $x \in E$ and $\{Tx_n\}_{n\geq 1}$ converges strongly to p, then $Tx = p$.

Definition 2.3 [6] A Banach space E is called to satisfy *Opial's condition* if for each sequence $\{x_n\}_{n\geq 1}$ in E which converges weakly to a point $x \in E$

$$\liminf_{n\to\infty}\|x_n - x\| < \liminf_{n\to\infty}\|x_n - y\|, \quad \forall y \in E - \{x\}.$$

Definition 2.4 Let K be a nonempty convex subset of a normed linear space E and $S, T : K \to K$ be two mappings. For an arbitrary $x_1 \in K$, the *modified three-step iteration sequence* with errors $\{x_n\}_{n\geq 1}$ with respect to S and T is defined by

$$\begin{aligned} z_n &= a''_n S x_n + b''_n T^n x_n + c''_n w_n, \\ y_n &= a'_n S x_n + b'_n T^n z_n + c'_n v_n, \\ x_{n+1} &= a_n S x_n + b_n T^n y_n + c_n u_n, \quad \forall n \geq 1, \end{aligned} \tag{2.1}$$

where $\{u_n\}_{n\geq 1}$, $\{v_n\}_{n\geq 1}$ and $\{w_n\}_{n\geq 1}$ are bounded sequences in K, $\{a_n\}_{n\geq 1}$, $\{b_n\}_{n\geq 1}$, $\{c_n\}_{n\geq 1}$, $\{a'_n\}_{n\geq 1}$, $\{b'_n\}_{n\geq 1}$, $\{c'_n\}_{n\geq 1}$, $\{a''_n\}_{n\geq 1}$, $\{b''_n\}_{n\geq 1}$ and $\{c''_n\}_{n\geq 1}$ are sequences in $[0,1]$ satisfying

$$a_n + b_n + c_n = a'_n + b'_n + c'_n = a''_n + b''_n + c''_n = 1, \quad \forall n \geq 1. \tag{2.2}$$

Remark 2.1 In case $S = I$ and $b''_n = c''_n = 0$ for $n \geq 1$, then the sequence $\{x_n\}_{n\geq 1}$ generated in (2.1) reduces to the usual modified Ishikawa sequence with errors.

Lemma 2.1 [4] *Let E be a Banach space satisfying Opial's condition and K be a nonempty closed convex subset of E. If $T : K \to K$ is an asymptotically nonexpansive mapping, then $I - T$ is demiclosed at zero.*

Lemma 2.2 [10] *Let E be a uniformly convex Banach space, $\{t_n\}_{n\geq 1} \subseteq [b,c] \subset (0,1)$, $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$ be sequences in E. If $\limsup_{n\to\infty}\|x_n\| \leq a$, $\limsup_{n\to\infty}\|y_n\| \leq a$ and $\lim_{n\to\infty}\|t_n x_n + (1-t_n)y_n\| = a$ for some constant $a \geq 0$, then $\lim_{n\to\infty}\|x_n - y_n\| = 0$.*

Lemma 2.3 [2] *Let E be a normed linear space. Then*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad \forall x, y \in E,\ j(x+y) \in J(x+y).$$

Lemma 2.4 [11] *Let $p > 1$ and $r > 0$ be two constants. Then a Banach space E is uniformly convex if and only if there exists a continuous and strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|ax + (1-a)y\|^p \le a\|x\|^p + (1-a)\|y\|^p - w_p(a)g(\|x-y\|)$$

for each $x, y \in B(\theta, r) = \{x : \|x\| \le r \text{ and } x \in E\}$, $a \in [0,1]$ and

$$w_p(a) = a^p(1-a) + a(1-a)^p$$

Lemma 2.5 [7] *Let $\{a_n\}_{n\ge1}$, $\{b_n\}_{n\ge1}$ and $\{c_n\}_{n\ge1}$ be sequences of nonnegative numbers satisfying the inequality*

$$a_{n+1} \le (1+c_n)a_n + b_n, \quad \forall n \ge 1.$$

If $\sum_{n=1}^{\infty} c_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. In particular, if $\{a_n\}_{n\ge1}$ has a subsequence which converges to zero, then $\lim_{n\to\infty} a_n = 0$.

3 Main Results

Lemma 3.1 *Let K be a nonempty convex subset of a normed linear space E. Let $S : K \to K$ be a mapping and $T : K \to K$ be uniformly L-Lipschitzian. Then*

$$\begin{aligned}\|x_{n+1} - Tx_{n+1}\| \le \|x_{n+1} - T^{n+1}x_{n+1}\| + L^2(L^2+2L+2)\|x_n - T^n x_n\| \\ + L(L+1)[(L^2+L+1)\|Sx_n - x_n\| + c_n\|u_n - x_n\| \\ + b_n c_n' L\|v_n - x_n\| + b_n b_n' c_n'' L^2\|w_n - x_n\|]\end{aligned}$$

for $n \ge 1$, where $\{x_n\}_{n\ge1}$ is defined by (2.1).

Proof Set $A_{n+1} = \|x_{n+1} - T^{n+1}x_n\|$, $B_{n+1} = \|Sx_{n+1} - x_{n+1}\|$ for $n \ge 1$. It follows that

$$\|z_n - x_n\| \le a_n''\|Sx_n - x_n\| + b_n''\|T^n x_n - x_n\| + c_n''\|w_n - x_n\|, \tag{3.1}$$

$$\begin{aligned}\|y_n - x_n\| &\le a_n'\|Sx_n - x_n\| + b_n'(L\|z_n - x_n\| + \|T^n x_n - x_n\|) + c_n'\|v_n - x_n\| \\ &\le a_n' B_n + b_n' L\|z_n - x_n\| + b_n' A_n + c_n'\|v_n - x_n\| \end{aligned}\tag{3.2}$$

and

$$\begin{aligned}\|x_{n+1} - Tx_{n+1}\| &\le \|x_{n+1} - T^{n+1}x_{n+1}\| + \|T^{n+1}x_{n+1} - Tx_{n+1}\| \\ &\le A_{n+1} + L\|T^n x_{n+1} - x_{n+1}\| \\ &\le A_{n+1} + L(\|T^n x_{n+1} - T^n x_n\| + \|T^n x_n - x_{n+1}\|) \\ &\le A_{n+1} + L^2\|x_{n+1} - x_n\| + L\|T^n x_n - x_{n+1}\| \\ &\le A_{n+1} + L^2 a_n B_n + L^2 b_n(\|T^n y_n - T^n x_n\| \\ &\quad + \|T^n x_n - x_n\|) + L^2 c_n\|u_n - x_n\| + La_n B_n + La_n A_n \\ &\quad + b_n L^2\|y_n - x_n\| + Lc_n A_n + Lc_n\|u_n - x_n\| \\ &\le A_{n+1} + L(L+1)a_n B_n + L(Lb_n + a_n + c_n)A_n \\ &\quad + L^2 b_n(L+1)\|y_n - x_n\| + Lc_n(L+1)\|u_n - x_n\| \end{aligned}\tag{3.3}$$

for $n \geq 1$. Substituting (3.1) and (3.2) into (3.3), we obtain that

$$\|x_{n+1} - Tx_{n+1}\| \leq A_{n+1} + L^2(L^2 + 2L + 2)A_n + L(L+1)[(L^2 + L + 1)B_n + c_n\|u_n - x_n\| + b_n c'_n L\|v_n - x_n\| + b_n b'_n c''_n L^2\|w_n - x_n\|]$$

for $n \geq 1$. This completes the proof of Lemma 2.1. □

Remark 3.1 Lemma 1.2 in [7], Lemma 3.1 in [5], Lemma 1.4 in [8] and Lemma 1.4 in [10] are special cases of Lemma 3.1.

Lemma 3.2 *Let K be a nonempty convex subset of a normed linear space E. Let $S : K \to K$ be a nonexpansive mapping and $T : K \to K$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subseteq [1, \infty)$ satisfying* $\lim_{n\to\infty} k_n = 1$ *and $F(S,T) \neq \emptyset$. If the following conditions*

$$\sum_{n=1}^{\infty}(k_n - 1) < \infty \tag{3.4}$$

and

$$\sum_{n=1}^{\infty} b_n b'_n c''_n < \infty, \quad \sum_{n=1}^{\infty} b_n c'_n < \infty, \quad \sum_{n=1}^{\infty} c_n < \infty \tag{3.5}$$

hold, then $\lim_{n\to\infty} \|x_n - q\|$ *exists for any $q \in F(S,T)$, where $\{x_n\}_{n\geq 1}$ is defined by (2.1).*

Proof Let $q \in F(S,T)$ and $L = \sup\{k_n : n \geq 1\}$. Note that $\{u_n - q\}_{n\geq 1}$, $\{v_n - q\}_{n\geq 1}$ and $\{w_n - q\}_{n\geq 1}$ are bounded. It follows that $M = \sup\{\|u_n - q\|, \|v_n - q\|, \|w_n - q\| : n \geq 1\} < \infty$. Since S is nonexpansive and T is asymptotically nonexpansive, by (2.1) we know that

$$\begin{aligned}
&\|x_{n+1} - q\| = \|a_n S x_n + b_n T^n y_n + c_n u_n - q\| \\
&\quad \leq a_n\|x_n - q\| + b_n k_n\|y_n - q\| + c_n\|u_n - q\| \\
&\quad \leq a_n\|x_n - q\| + b_n k_n(a'_n\|x_n - q\| + b'_n k_n\|z_n - q\| + c'_n\|v_n - q\|) + c_n M \\
&\quad \leq (a_n + b_n k_n a'_n)\|x_n - q\| + b_n b'_n k_n^2(a''_n\|x_n - q\| + b''_n k_n\|x_n - q\| \\
&\qquad + c''_n\|w_n - q\|) + b_n c'_n k_n M + c_n M \\
&\quad \leq [a_n + b_n k_n a'_n + b_n b'_n k_n^2(a''_n + b''_n k_n)]\|x_n - q\| \\
&\qquad + (b_n b'_n c''_n k_n + b_n c'_n k_n M + c_n)M \\
&\quad \leq [1 - b_n + b_n k_n(1 - b'_n) + b_n b'_n k_n^2(1 - b''_n + b''_n k_n)]\|x_n - q\| \\
&\qquad + (L b_n b'_n c''_n + LM b_n c'_n + c_n)M \\
&\quad \leq [1 + b_n(k_n - 1)(1 + L + L^2)]\|x_n - q\| \\
&\qquad + (L b_n b'_n c''_n + LM b_n c'_n + c_n)M
\end{aligned} \tag{3.6}$$

for $n \geq 1$. It follows from Lemma 2.5, (3.4) and (3.5) that $\lim_{n\to\infty} \|x_n - q\|$ exists. This completes the proof. □

Remark 3.2 Lemma 3.2 generalizes Lemma 3.2 in [5], Lemma 3 in [7] and Lemma 1.2 in [10].

Lemma 3.3 *Let K be a nonempty convex subset of a uniformly convex Banach space E. Let $S : K \to K$ be a nonexpansive mapping and $T : K \to K$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subseteq [1,\infty)$ satisfying (3.4), $\lim_{n\to\infty} k_n = 1$, $F(S,T) \neq \emptyset$ and*

$$\|x - Ty\| \le \|Sx - Ty\|, \quad \forall x, y \in K. \tag{3.7}$$

Suppose that

$$\sum_{n=1}^{\infty} c'_n < \infty, \quad \sum_{n=1}^{\infty} b'_n c''_n < \infty, \quad \sum_{n=1}^{\infty} c_n < \infty, \tag{3.8}$$

$$(1 + \limsup_{n\to\infty} b''_n) \cdot \limsup_{n\to\infty} b'_n < 1, \tag{3.9}$$

$$0 < a \le b_n \le b < 1, \quad \forall n \ge 1, \tag{3.10}$$

where a and b are constants. Then $\lim_{n\to\infty} \|x_n - Sx_n\| = \lim_{n\to\infty} \|x_n - Tx_n\| = 0$, where $\{x_n\}_{n\ge1}$ is defined by (2.1).

Proof Let $q \in F(S,T)$. Lemma 3.2 ensures that $\lim_{n\to\infty} \|x_n - q\|$ exists. Set $\lim_{n\to\infty} \|x_n - q\| = d$. Since $\{u_n\}_{n\ge1}$, $\{v_n\}_{n\ge1}$ and $\{w_n\}_{n\ge1}$ are bounded sequences, it follows that

$$M = \sup\{\|u_n - q\|, \|v_n - q\|, \|x_n - v_n\|, \|x_n - w_n\|, \|x_n - u_n\| : n \ge 1\} < \infty.$$

Observe that

$$\begin{aligned}\lim_{n\to\infty} \|x_{n+1} - q\| &= \lim_{n\to\infty} \|(1 - b_n - c_n)Sx_n + b_n T^n y_n + c_n u_n - q\| \\ &= \lim_{n\to\infty} \|(1 - b_n)[Sx_n - q - c_n(Sx_n - u_n)] \\ &\quad + b_n[T^n y_n - q - c_n(Sx_n - u_n)]\|. \end{aligned} \tag{3.11}$$

From the nonexpansivity of S and (3.8), we deduce that

$$\begin{aligned}&\limsup_{n\to\infty} \|Sx_n - q - c_n(Sx_n - u_n)\| \\ &\le \limsup_{n\to\infty}(\|x_n - q\| + c_n\|x_n - q\| + c_n\|u_n - q\|) \\ &\le \limsup_{n\to\infty}[(1 + c_n)\|x_n - q\| + c_n M] \le d. \end{aligned} \tag{3.12}$$

Since S is nonexpansive and T is asymptotically nonexpansive, by (2.1) we derive that

$$\begin{aligned}&\|T^n y_n - q - c_n(Sx_n - u_n)\| \\ &\le k_n\|y_n - q\| + c_n\|Sx_n - q\| + c_n\|u_n - q\| \\ &\le k_n[a'_n\|x_n - q\| + b'_n k_n\|z_n - q\|] + (c'_n k_n + c_n)M + c_n\|x_n - q\| \\ &\le (a'_n k_n + c_n)\|x_n - q\| + b'_n k_n^2\|z_n - q\| + (c'_n k_n + c_n)M \\ &\le [a'_n k_n + c_n + b'_n k_n^2(a''_n + b''_n k_n)]\|x_n - q\| + (c'_n k_n + c_n + b'_n c''_n k_n^2)M \\ &\le [k_n + b'_n k_n(k_n - 1) + b'_n k_n^2 b''_n(k_n - 1) + c_n]\|x_n - q\| \\ &\quad + (c'_n k_n + c_n + b'_n c''_n k_n^2)M \end{aligned} \tag{3.13}$$

for $n \geq 1$. In view of (3.4), (3.8) and (3.13), we conclude that

$$\limsup_{n\to\infty} \|T^n y_n - q - c_n(Sx_n - u_n)\| \leq d. \tag{3.14}$$

On account of (3.10)–(3.12), (3.14) and Lemma 2.2, we see that

$$\begin{aligned}
&\lim_{n\to\infty} \|Sx_n - T^n y_n\| = \\
&= \lim_{n\to\infty} \|[Sx_n - q - c_n(Sx_n - u_n)] - [T^n y_n - q - c_n(Sx_n - u_n)]\| \\
&= 0,
\end{aligned} \tag{3.15}$$

which implies that

$$\lim_{n\to\infty} \|x_n - T^n y_n\| = 0 \tag{3.16}$$

by (3.7). Notice that

$$\|Sx_n - x_n\| \leq \|Sx_n - T^n y_n\| + \|x_n - T^n y_n\|, \quad \forall n \geq 1.$$

Thus (3.15) and (3.16) mean that

$$\lim_{n\to\infty} \|Sx_n - x_n\| = 0. \tag{3.17}$$

It is easy to verify that

$$\begin{aligned}
\|x_n - T^n x_n\| &\leq \|x_n - T^n y_n\| + k_n\|y_n - x_n\| \\
&\leq \|x_n - T^n y_n\| + k_n[a'_n\|Sx_n - x_n\| + b'_n k_n\|z_n - x_n\| \\
&\quad + b'_n\|T^n x_n - x_n\| + c'_n M] \\
&\leq \|x_n - T^n y_n\| + k_n(a'_n + k_n b'_n a''_n)\|Sx_n - x_n\| \\
&\quad + k_n b'_n(1 + k_n b''_n)\|T^n x_n - x_n\| + k_n(c'_n + k_n b'_n c''_n)M
\end{aligned} \tag{3.18}$$

for $n \geq 1$. Note that (3.9) implies that there exists a positive integer N satisfying $k_n b'_n(1 + k_n b''_n) < 1$ for $n \geq N$. It follows from (3.18) that

$$\begin{aligned}
\|x_n - T^n x_n\| &\leq \frac{1}{1 - k_n b'_n(1 + k_n b''_n)} [\|x_n - T^n y_n\| \\
&\quad + k_n(a'_n + k_n b'_n a''_n)\|Sx_n - x_n\| + k_n(c'_n + k_n b'_n c''_n)M]
\end{aligned} \tag{3.19}$$

for $n \geq N$. According to (3.8), (3.9), (3.16), (3.17) and (3.19), we conclude that

$$\lim_{n\to\infty} \|x_n - T^n x_n\| = 0. \tag{3.20}$$

In terms of (3.8), (3.17), (3.20) and Lemma 3.1, we get that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

This completes the proof. □

Remark 3.3 Lemma 3.3 extends Lemma 3.3 in [6], Lemma 4 in [8] and Theorem 1 in [9].

Theorem 3.1 *Let E be a uniformly convex Banach space satisfying Opial's condition and K be a nonempty closed convex subset of E. Let $S : K \to K$ be a nonexpansive mapping and $T : K \to K$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subseteq [1, \infty)$ such that $\lim_{n\to\infty} k_n = 1$ and $F(S,T) \neq \emptyset$. If (3.4) and (3.7)–(3.10) hold, then the modified three-step iteration sequences with errors $\{x_n\}_{n\geq 1}$ with respect to S and T defined by (2.1) converges weakly to a common fixed point of S and T.*

Proof It follows from Lemma 3.2 that $\{x_n\}_{n\geq1}$ is bounded. Hence $\{x_n\}_{n\geq1}$ has a subsequence $\{x_{n_j}\}_{j\geq1}$, which converges weakly to p. Since $\{x_{n_j}\}_{j\geq1} \subseteq K$ and K is weakly closed, it follows that $p \in K$. From Lemmas 3.3 and 2.1 we deduce that $I - T$ and $I - S$ are demiclosed at zero. Hence $(I - T)p = (I - S)p = 0$. That is, $p \in F(S,T)$. Suppose that $\{x_n\}_{n\geq1}$ does not converge weakly to p . Then there exists another subsequence $\{x_{m_k}\}_{k\geq1}$ of $\{x_n\}_{n\geq1}$ which converges weakly to some $q \neq p$. It is clear that $q \in F(S,T)$, $\lim_{n\to\infty} \|x_n - p\|$ and $\lim_{n\to\infty} \|x_n - q\|$ exist. Let $a = \lim_{n\to\infty} \|x_n - p\|$, $b = \lim_{n\to\infty} \|x_n - q\|$. Because E satisfies Opial's condition, we obtain that

$$\begin{aligned} a &= \liminf_{j\to\infty} \|x_{n_j} - p\| < \liminf_{j\to\infty} \|x_{n_j} - q\| \\ &= b = \liminf_{k\to\infty} \|x_{m_k} - q\| < \liminf_{k\to\infty} \|x_{m_k} - p\| = a, \end{aligned}$$

which is a contradiction. Hence $p = q$ and $\{x_n\}_{n\geq1}$ converges weakly to $p \in F(S,T)$. This completes the proof. □

Lemma 3.4 *Let K be a nonempty bounded closed convex subset of a normed linear space E. Let $S : K \to K$ be a mapping and $T : K \to K$ be uniformly L-Lipschitzian. Then*

$$\begin{aligned} \|x_{n+1} - Tx_{n+1}\| \leq\ & \|x_{n+1} - T^{n+1}x_{n+1}\| + L\|x_n - T^n x_n\| \\ & + L(L+1)[(1 + L + L^2)(\|x_n - T^n x_n\| + \|Sx_n - x_n\|) \\ & + c_n\|u_n - Sx_n\| + Lb_n c'_n\|v_n - Sx_n\| \\ & + b_n b'_n c''_n L^2\|w_n - Sx_n\|] \end{aligned} \tag{3.21}$$

for $n \geq 1$,where $\{x_n\}_{n\geq1}$ is defined by (2.1).

Proof Put

$$A_n = c_n(u_n - Sx_n),\ B_n = c'_n(v_n - Sx_n),\ C_n = c''_n(w_n - Sx_n),\ \forall n \geq 1. \tag{3.22}$$

Then the sequence $\{x_n\}_{n\geq1}$ defined by (2.1) can be rewritten as

$$\begin{aligned} z_n &= (1 - b''_n)Sx_n + b''_n T^n x_n + C_n, \\ y_n &= (1 - b'_n)Sx_n + b'_n T^n z_n + B_n, \\ x_{n+1} &= (1 - b_n)Sx_n + b_n T^n y_n + A_n, \quad \forall n \geq 1. \end{aligned} \tag{3.23}$$

The rest of the proof is exactly the same as that of Lemma 3.1, and is omitted. This completes the proof. □

Remark 3.4 Lemma 3.4 is an improvement of Lemma 3 in [1] and Lemma 1.2 in [9].

Lemma 3.5 *Let K be a nonempty bounded closed convex subset of a real Banach space E. Let $S : K \to K$ be a nonexpansive mapping and $T : K \to K$ be a uniformly L-Lipschitzian and asymptotically nonexpansive mapping with a sequence $\{k_n\} \subseteq [1, \infty)$ satisfying $\lim_{n\to\infty} k_n = 1$, $F(S,T) \neq \emptyset$, (3.4) and (3.7). Suppose that (3.8), (3.10) and*

$$(1 + L \limsup_{n\to\infty} b''_n) \cdot \limsup_{n\to\infty} b'_n < L^{-1} \tag{3.24}$$

hold. Then $\lim_{n\to\infty} \|Sx_n - x_n\| = \lim_{n\to\infty} \|x_n - T^n x_n\| = 0$, where $\{x_n\}_{n\geq1}$ is defined by (2.1).

Proof Let $\{A_n\}_{n\geq1}$, $\{B_n\}_{n\geq1}$, $\{C_n\}_{n\geq1}$ be defined by (3.22) and $q \in F(S,T)$. Note that K is a nonempty bounded closed convex subset and $\{x_n\}_{n\geq1}$, $\{y_n\}_{n\geq1}$, $\{z_n\}_{n\geq1}$, $\{T^n x_n\}_{n\geq1}$, $\{T^n y_n\}_{n\geq1}$, $\{T^n z_n\}_{n\geq1}$, $\{Sx_n\}_{n\geq1}$ are in K. Then there exists $r > 0$ such that

$$\begin{aligned} K \cup \{&x_n - q, y_n - q, z_n - q, Sx_n - q, Sx_n - u_n, Sx_n - v_n, Sx_n - w_n, \\ &Sx_n - q + A_n, Sx_n - q + B_n, Sx_n - q + C_n, T^n x_n - q + A_n, \\ &T^n y_n - q + B_n, T^n z_n - q + C_n, T^n y_n - q + A_n, T^n y_n - q + C_n\} \\ &\subset B(\theta, r) \end{aligned}$$

for any $n \geq 1$. From Lemma 2.3 we get that

$$\begin{aligned} \|Sx_n - q + A_n\|^2 &\leq \|Sx_n - q\|^2 + 2\langle A_n, j(Sx_n - q + A_n)\rangle \\ &\leq \|x_n - q\|^2 + 2\|A_n\| \cdot \|Sx_n - q + A_n\| \\ &\leq \|x_n - q\|^2 + 2r\|A_n\| \end{aligned} \tag{3.25}$$

for $j(Sx_n - q + A_n) \in J(Sx_n - q + A_n)$ and $n \geq 1$. Similarly we have

$$\|T^n y_n - q + A_n\|^2 \leq \|T^n y_n - q\|^2 + 2r\|A_n\| \leq k_n^2 \|y_n - q\|^2 + 2r\|A_n\| \tag{3.26}$$

for $n \ge 1$. It follows from (3.25), (3.26) and Lemma 2.4 that

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \|(1-b_n)(Sx_n - q + A_n) + b_n(T^n y_n - q + A_n)\|^2 \\
&\le (1-b_n)\|Sx_n - q + A_n\|^2 + b_n(T^n y_n - q + A_n)\|^2 \\
&\quad - w_2(b_n)g(\|Sx_n - T^n y_n\|) \\
&\le (1-b_n)(\|x_n - q\|^2 + 2r\|A_n\|) + b_n(\|T^n y_n - q\|^2 + 2r\|A_n\|) \\
&\quad - b_n(1-b_n)g(\|Sx_n - T^n y_n\|) \\
&= \|x_n - q\|^2 + b_n(\|T^n y_n - q\|^2 - \|y_n - q\|^2) \\
&\quad + b_n(\|y_n - q\|^2 - \|x_n - q\|^2) + 2r\|A_n\| \\
&\quad - b_n(1-b_n)g(\|Sx_n - T^n y_n\|) \\
&\le \|x_n - q\|^2 + b_n(k_n^2 - 1)\|y_n - q\|^2 + b_n(\|y_n - q\|^2 - \|x_n - q\|^2) \\
&\quad + 2r\|A_n\| - b_n(1-b_n)g(\|Sx_n - T^n y_n\|) \qquad (3.27)
\end{aligned}$$

for $n \ge 1$. Obviously we have

$$\begin{aligned}
&\|z_n - q\|^2 - \|x_n - q\|^2 \\
&\le (1-b_n'')\|x_n - q\|^2 + b_n''\|T^n x_n - q\|^2 - \|x_n - q\|^2 + 2r\|C_n\| \\
&\le b_n''(k_n^2 - 1)\|x_n - q\|^2 + 2r\|C_n\| \qquad (3.28)
\end{aligned}$$

and

$$\begin{aligned}
&\|y_n - q\|^2 - \|x_n - q\|^2 \\
&\le (1-b_n')\|x_n - q\|^2 + b_n'\|T^n z_n - q\|^2 - \|x_n - q\|^2 + 2r\|B_n\| \\
&\le b_n'(k_n^2 - 1)\|z_n - q\|^2 + b_n'(\|z_n - q\|^2 - \|x_n - q\|^2) + 2r\|B_n\| \qquad (3.29)
\end{aligned}$$

for $n \ge 1$. Using (3.27)–(3.29) we obtain that

$$\begin{aligned}
&\|x_{n+1} - q\|^2 \\
&\le \|x_n - q\|^2 + b_n(k_n^2 - 1)\|y_n - q\|^2 + b_n b_n'(k_n^2 - 1)\|z_n - q\|^2 \\
&\quad + b_n b_n' b_n''(k_n^2 - 1)\|x_n - q\|^2 + 2rb_n b_n'\|C_n\| + 2rb_n\|B_n\| + 2r\|A_n\| \\
&\quad - b_n(1-b_n)g(\|Sx_n - T^n y_n\|) \\
&\le \|x_n - q\|^2 + b_n(k_n^2 - 1)[\|y_n - q\|^2 + b_n'\|z_n - q\|^2 \\
&\quad + b_n' b_n''\|x_n - q\|^2] + 2r(b_n b_n'\|C_n\| + b_n\|B_n\| + \|A_n\|) \\
&\quad - b_n(1-b_n)g(\|Sx_n - T^n y_n\|) \qquad (3.30)
\end{aligned}$$

for $n \ge 1$. Since $\{x_n - q\}_{n\ge1}$, $\{y_n - q\}_{n\ge1}$ and $\{z_n - q\}_{n\ge1}$ belong to $B(\theta, r)$, (3.10) and (3.30) ensure that

$$\begin{aligned}
\|x_{n+1} - q\|^2 &\le \|x_n - q\|^2 + 3r^2(1 + \sup\{k_n : n \ge 1\})(k_n - 1) \\
&\quad + 2r^2(b_n b_n' c_n'' + b_n c_n' + c_n) - a(1-b)g(\|Sx_n - T^n y_n\|) \qquad (3.31)
\end{aligned}$$

for $n \ge 1$. Therefore

$$\begin{aligned}
a(1-b)g(\|Sx_n - T^n y_n\|) &\le \|x_n - q\|^2 - \|x_{n+1} - q\|^2 \\
+ 3r^2(1 + \sup\{k_n : n \ge 1\})&(k_n - 1) + 2r^2(b_n b_n' c_n'' + b_n c_n' + c_n)
\end{aligned}$$

for $n \geq 1$. This yields that

$$a(1-b)\sum_{n=1}^{m} g(\|Sx_n - T^n y_n\|) \leq \|x_1 - q\|^2$$
$$+ 3r^2(1+\sup\{k_n : n \geq 1\})\sum_{n=1}^{m}(k_n - 1) + 2r^2\sum_{n=1}^{m}(b_n b'_n c''_n + b_n c'_n + c_n)$$

for $m \geq 1$. Letting $m \to \infty$ in the above inequality, we derive that

$$\sum_{n=1}^{\infty} g(\|Sx_n - T^n y_n\|) < \infty,$$

which implies that

$$\lim_{n\to\infty} g(\|Sx_n - T^n y_n\|) = 0. \tag{3.32}$$

Note that $g : [0,\infty) \to [0,\infty)$ is continuous and strictly increasing with $g(0) = 0$. It follows from (3.32) that

$$\lim_{n\to\infty} \|Sx_n - T^n y_n\| = 0. \tag{3.33}$$

On account of (3.7) and (3.33), we know that

$$\lim_{n\to\infty} \|x_n - T^n y_n\| = 0. \tag{3.34}$$

It follows from (3.33) and (3.34)that

$$\lim_{n\to\infty} \|Sx_n - x_n\| = 0. \tag{3.35}$$

By virtue of (3.23) we have

$$\begin{aligned}
&\|x_n - y_n\| \\
&\leq \|x_n - T^n y_n\| + \|T^n y_n - y_n\| \\
&\leq \|x_n - T^n y_n\| + (1-b'_n)\|Sx_n - T^n y_n\| + b'_n L\|z_n - y_n\| + c'_n r \\
&\leq \|x_n - T^n y_n\| + (1-b'_n)\|Sx_n - T^n y_n\| \\
&\quad + b'_n L(\|z_n - x_n\| + \|x_n - y_n\|) + c'_n r \\
&\leq \|x_n - T^n y_n\| + (1-b'_n)\|Sx_n - T^n y_n\| + b'_n L[(1-b''_n)\|Sx_n - x_n\| \\
&\quad + b''_n\|T^n x_n - x_n\| + c''_n r + \|x_n - y_n\|] + c'_n r
\end{aligned} \tag{3.36}$$

for $n \geq 1$. Notice that (3.24) ensures that there exists a positive integer M satisfying

$$b'_n < L^{-1} \quad \text{and} \quad (1+Lb''_n)b'_n < L^{-1} \quad \text{for } n \geq M. \tag{3.37}$$

From (3.36) and (3.37), we conclude that

$$\|x_n - y_n\| \leq \frac{1}{1-b'_n L}[\|x_n - T^n y_n\| + \|Sx_n - T^n y_n\| + b'_n L(\|Sx_n - x_n\|$$
$$+ b''_n\|T^n x_n - x_n\| + rc''_n) + c'_n r]$$

and

$$\begin{aligned}\|T^nx_n - x_n\| \ &\le \ \|T^nx_n - T^ny_n\| + \|x_n - T^ny_n\| \le L\|x_n - y_n\| + \|x_n - T^ny_n\| \\ &\le \ L \cdot \frac{1}{1-b_n'L}\{\|x_n - T^ny_n\| + \|Sx_n - T^ny_n\| \\ &\quad + b_n'L[\|Sx_n - x_n\| + b_n''\|T^nx_n - x_n\| + rc_n''] + c_n'r\} + \|x_n - T^ny_n\|\end{aligned}$$

for $n \ge M$. Simplifying we get that

$$\begin{aligned}\|T^nx_n - x_n\| \ \le \ &\frac{1}{1-b_n'L-b_n'b_n''L^2}[(L+1)\|x_n - T^ny_n\| + L\|Sx_n - T^ny_n\| \\ &+ b_n'L^2\|Sx_n - x_n\| + L(b_n'Lc_n'' + c_n')r] \qquad (3.38)\end{aligned}$$

for $n \ge M$. It follows from (3.8), (3.10), (3.33)–(3.35) and (3.38) that

$$\lim_{n\to\infty} \|T^nx_n - x_n\| = 0.$$

This completes the proof. □

Remark 3.5 Lemma 3.5 improves Lemma 4 in [1], Theorem 1 in [9] and Lemma 1.4 in [10].

Theorem 3.2 *Let E be a real uniformly convex Banach space, K be a nonempty bounded closed convex subset of E. Let $S : K \to K$ be a nonexpansive mapping and $T : K \to K$ be a uniformly L-Lipschitzian and asymptotically nonexpansive mapping with a sequence $\{k_n\} \subseteq [1,\infty)$ satisfying $\lim_{n\to\infty} k_n = 1$, $F(S,T) \ne \emptyset$, (3.4) and (3.7). Suppose that (3.8), (3.10) and (3.24) hold. If T is semi-compact, then the modified three-step iteration sequences with errors $\{x_n\}_{n\ge1}$ with respect to S and T defined by* (2.1) *converges strongly to a common fixed point of S and T.*

Proof It follows from Lemmas 3.4 and 3.5 and (3.8) that $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. Since T is semi-compact, there exists a subsequence $\{x_{n_i}\}_{i\ge1} \subset \{x_n\}_{n\ge1}$ such that $x_{n_i} \to q \in K$ as $i \to \infty$. By the continuity of S and T, (3.8) and Lemma 3.5, we conclude that

$$\lim_{i\to\infty} \|Tx_{n_i} - x_{n_i}\| = \|q - Tq\| = 0, \qquad \lim_{i\to\infty} \|Sx_{n_i} - x_{n_i}\| = \|q - Sq\| = 0.$$

That is, q is a common fixed point of S and T in K. From (3.31) we know that

$$\begin{aligned}\|x_{n+1} - q\|^2 \ \le \ &\|x_n - q\|^2 + 3r^2(1 + \sup\{k_n : n \ge 1\})(k_n - 1) \\ &+ 2r^2(b_nb_n'c_n'' + b_nc_n' + c_n) \qquad (3.39)\end{aligned}$$

for $n \ge 1$. Then (3.4), (3.8), (3.39) and Lemma 2.5 guarantee that $\lim_{n\to\infty} \|x_n - q\|^2 = 0$. That is $\lim_{n\to\infty} x_n = q$. This completes the proof. □

Remark 3.6 Theorems 3.1 and 3.2 extend, improve and unify Theorems 1.1 and 1.2 in [1], Theorem 3.1 in [5], Theorems 1 and 2 in [7], Theorems 2 and 3 in [8], Theorem 1.5 in [9] and Theorems 2.1 and 2.2 in [10] and in the following ways:

(1) the identity mapping in [1], [5], [7]–[10] is replaced by the more general nonexpansive mapping.

(2) the usual modified Mann iteration methods in [10], the usual modified Ishikawa iteration methods in [8] and [9], the usual modified Ishikawa iterations methods with errors in [1] and [7] and the usual modified three-step iteration methods with errors in [5] are extended to the modified three-step iteration methods with errors with respect to a pair of mappings.

(3) the conditions (3.8) and (3.10) are weaker than the conditions $\lim_{n\to\infty} b_n = 0$ and $0 < \epsilon \le a_n \le 1-\epsilon$, for all $n \ge 1$, imposed on Theorems 1.1 and 1.2 in [1], Theorem 1 in [7], Theorems 2 and 3 in [8] and Theorem 1.5 in [9].

Remark 3.7 We would like to point out that $\sum_{n=1}^{\infty}(k_n^p - 1) < \infty$ in [9] and $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$ in [1] and [10] are equivalent to condition (3.4).

The following example shows that Theorems 3.1 and 3.2 extend substantially the corresponding results in [1], [5] and [7]–[10].

Example 3.1 Let E be the real line with the usual norm $|\cdot|$ and let $K = [0,1]$. Define S and $T : K \to K$ by

$$Tx = \begin{cases} -\sin x, & x \in [0,1], \\ \sin x, & x \in [-1,0) \end{cases} \quad \text{and} \quad Sx = \begin{cases} x, & x \in [0,1], \\ -x, & x \in [-1,0) \end{cases}$$

for $x \in K$. Obviously $F(S,T) = \{0\}$ and T is semi-compact. Now we check that T is nonexpansive. In fact, if x and $y \in [0,1]$ or x and $y \in [-1,0)$, then $|Tx - Ty| = |\sin x - \sin y| \le |x-y|$; if $x \in [0,1]$ and $y \in [-1,0)$ or $x \in [-1,0)$ and $y \in [0,1]$, then

$$|Tx - Ty| = |\sin x + \sin y| = 2\left|\sin\frac{x+y}{2}\cos\frac{x-y}{2}\right| \le |x+y| \le |x-y|.$$

That is, T is nonexpansive. Similarly we can verify that S is nonexpansive. Thus S is uniformly 1-Lispchitzian and asymptotically nonexpansive. In order to show that S and T satisfy (3.7), we have to consider the following cases:

Case 1. Suppose that x and $y \in [0,1]$. It follows that

$$|x - Ty| = |x + \sin y| = |Sx - Ty|;$$

Case 2. Suppose that x and $y \in [-1,0)$ Then we easily see that

$$|x - Ty| = |x - \sin y| \le |-x - \sin y| = |Sx - Ty|;$$

Case 3. Suppose that $x \in [-1,0)$ and $y \in [0,1]$. It is easy to verify that

$$|x - Ty| = |x + \sin y| \le |-x + \sin y| = |Sx - Ty|;$$

Case 4. Suppose that $x \in [0, 1]$ and $y \in [-1, 0)$. It follows that

$$|x - Ty| = |x - \sin y| = |Sx - Ty|.$$

Hence (3.7) is satisfied. Suppose that $\{u_n\}_{n\geq 1}$, $\{v_n\}_{n\geq 1}$ and $\{w_n\}_{n\geq 1}$ are arbitrary sequences in K,

$$a = \frac{3}{5}, \quad b = \frac{6}{7}, \quad a_n = \frac{2}{5} - \frac{1}{3n+2} - \frac{1}{6n^2},$$
$$a'_{2n} = 1 - \frac{1}{3n} - \frac{1}{2n^2+3,} \quad a'_{2n-1} = \frac{1}{2} - \frac{1}{3n+2} - \frac{1}{2n^2+3},$$
$$b_n = \frac{3}{5} + \frac{1}{3n+2,} \quad b'_{2n} = \frac{1}{3n}, \quad b'_{2n-1} = \frac{1}{3n+2} + \frac{1}{2},$$
$$c_n = \frac{1}{6n^2}, \quad c'_{2n} = c'_{2n-1} = \frac{1}{2n^2+3},$$
$$a''_n = \frac{3}{7} + \frac{1}{12n}, \quad b''_n = \frac{4}{7} - \frac{1}{12n} - \frac{1}{4n^2}, \quad c''_n = \frac{1}{4n^2}$$

for $n \geq 1$. Thus the conditions of Theorems 3.1 and 3.2 are fulfilled. Hence Theorems 3.1 and 3.2 guarantee that the modified Ishikawa iteration sequences with errors $\{x_n\}_{n\geq 1}$ with respect to S and T defined by (2.1) converges both strongly and weakly to 0, respectively. But the results in [1], [5] and [7]–[10] are not applicable.

References

[1] Chang, S. S.: *On the approximation problem of fixed points for asymptotically nonexpansive mappings.* Indian J. Pure Appl. Math. **32** (2001), 1297–1307.

[2] Chang, S. S.: *Some problems and results in the study of nonlinear analysis.* Nonlinear Anal. TMA **30** (1997), 4197–4208.

[3] Goebel, K., Kirk, W. A.: *A fixed point theorem for asymptotically nonexpansive mappings.* Proc. Amer. Math. Soc. **35** (1972), 171–174.

[4] Gornicki, J.: *Weak convergence theorems for asymptotically nonexpansive mappings in uniformly convex Banach spaces.* Comment. Math. Univ. Carolina **30** (1989), 249–252.

[5] Liu, Z., Kang, S. M.: *Weak and strong convergence for fixed points of asymptotically nonexpansive mappings.* Acta. Math. Sinica **20** (2004), 1009–1018.

[6] Opial, Z.: *Weak convergence of the sequence of successive approximations for nonexpansive mappings.* Bull. Amer. Math. Soc. **73** (1967), 591–597.

[7] Osilike, M. O., Aniagbosor, S. C.: *Weak and strong convergence theorems for fixed points of asymptotically nonexpansive mappings.* Math. Comput. Modelling **32** (2000), 1181–1191.

[8] Rhoades, B. E.: *Fixed point iteration for certain nonlinear mappings.* J. Math. Anal. Appl. **183** (1994), 118–120.

[9] Schu, J.: *Iterative construction of fixed points of asymptotically nonexpansive mappings.* J. Math. Anal. Appl. **158** (1991), 407–413.

[10] Schu, J.: *Weak and strong convergence of fixed points of asymptotically nonexpansive mappings.* Bull. Austral. Math. Soc. **43** (1991), 153–159.

[11] Xu, H. K.: *Inequalities in Banach spaces with applications.* Nonlinear Anal. TMA **16** (1991), 1127–1138.

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Fixed Point Analysis for Non-oscillatory Solutions of Quasi Linear Ordinary Differential Equations

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Abstract

The paper deals with the quasi-linear ordinary differential equation $(r(t)\varphi(u'))' + g(t,u) = 0$ with $t \in [0,\infty)$. We treat the case when g is not necessarily monotone in its second argument and assume usual conditions on $r(t)$ and $\varphi(u)$. We find necessary and sufficient conditions for the existence of unbounded non-oscillatory solutions. By means of a fixed point technique we investigate their growth, proving the coexistence of solutions with different asymptotic behaviors. The results generalize previous ones due to *Elbert–Kusano*, [Acta Math. Hung. 1990]. In some special cases we are able to show the exact asymptotic growth of these solutions. We apply previous analysis for studying the non-oscillatory problem associated to the equation when $\varphi(u) = u$. Several examples are included.

Key words: Quasi-linear second order equations; unbounded, oscillatory and non-oscillatory solutions; fixed-point techniques.

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1 Introduction

The paper deals with the quasi-linear ordinary differential equation

$$(r(t)\varphi(u'))' + g(t,u) = 0 \text{ on } [0,+\infty) \tag{1}$$

under the following assumptions concerning r, φ and g

$$\begin{aligned}
&r \in C[0,+\infty), r(t) > 0 \text{ for } t \in [0,+\infty);\\
&\varphi \in C(\mathbb{R}), \text{ strictly increasing, surjective, } v\varphi(v) > 0 \text{ for } v \neq 0;\\
&\textstyle\int_0^\infty \varphi^{-1}(\frac{k}{r(s)})\, ds = \infty \text{ for } k \neq 0;\\
&g(t,u) \in C([0,+\infty) \times \mathbb{R}) \text{ with } ug(t,u) > 0 \text{ for } u \neq 0 \text{ and } t \geq 0.
\end{aligned} \tag{2}$$

As usual by solution we shall mean a continuously differentiable function u such that $r(t)\varphi(u')$ has a continuous derivative satisfying (1). We recall that a solution of (1) is said to be oscillatory if it has an infinite sequence of zeros clustering at ∞, non-oscillatory otherwise. The oscillatory and non-oscillatory behavior of equation (1) is of special interest. On this purpose, it is important to find necessary and/or sufficient conditions for the existence of solutions with a prescribed asymptotic behavior. The following lemma gives the classification of all possible non-oscillatory solutions of (1) according to their asymptotic behavior. The result is due to Elbert–Kusano (see [6, Lemma 1]) and since its proof does not depend on the monotonicity of $g(t,\cdot)$, which is assumed in [6], it is also valid in this more general context.

Lemma 1 *[6, Lemma 1] Any non-oscillatory solution $u(t)$ of (1) is of one of the following types:*

I) $\lim_{t\to\infty} |u(t)| = \infty$ *and* $\lim_{t\to\infty} r(t)\varphi(u'(t)) = \text{const} \neq 0.$

II) $\lim_{t\to\infty} |u(t)| = \infty$ *and* $\lim_{t\to\infty} r(t)\varphi(u'(t)) = 0.$

III) $\lim_{t\to\infty} u(t) = \text{const} \neq 0$ *and* $\lim_{t\to\infty} r(t)\varphi(u'(t)) = 0.$

In Sections 2 and 3 we obtain sufficient and sometimes also necessary conditions for the existence of an unbounded non-oscillatory solution respectively of type I and II (see Theorems 1, 2 and 3). In Section 3 in some special cases, we also discuss the coexistence of type I and II solutions and prove the exact asymptotic behavior of a type II solution (see Proposition 2). Our main investigation technique combine a linearization device with Schauder–Tychonoff fixed point theorem. We compare our results with previous ones, in particular with those in [6] and furnish several examples. Finally, Section 4 deals with the special case

$$(r(t)u')' + g(t,u) = 0, \quad t \in [0,\infty) \tag{3}$$

occurring when $\varphi(u) = u$. Applying previous analysis we discuss the non-oscillatory properties of (3).

Equation (1) arises in several applications. We quote, as an example, the important study of the polar form of the semi-linear elliptic partial differential equation $\operatorname{div}(|Du|^{\alpha-2}Du) + q(t)f(u) = 0$. When $f(u) = |u|^{\gamma-1}u$, this reduces to the investigation of $(|u'|^{\alpha-1}u')' + q(t)|u|^{\gamma-1}u = 0$, including the half-linear equation ($\alpha = \gamma$) and the generalized Emden–Fowler equation ($\alpha = 1$, $q(t) = (t+1)^{-m}$). Therefore, a wide literature is available, concerning the existence and the asymptotic behavior of the solutions of (1) as well as their oscillatory properties. See e.g. [1]–[4], [6]–[11], and references therein contained. However, most of the quoted papers deals with the case when $g(t,u) = q(t)f(u)$ and very often it is assumed $f(u) = |u|^{\gamma-1}u$ for some $\gamma > 0$. In addition, also when $g(t,u)$ has not separable variables, as in [6] and [10], $g(t,\cdot)$ is always increasing. The main purpose of this paper is to investigate these matters in the case when g is not necessarily monotone in its second argument. More precisely, we often assume the existence of a constant $L > 0$ such that

$$|g(t,v)| \le L|g(t,u)| \quad \text{for } u \in \mathbb{R},\ v \in [\min\{0,u\}, \max\{0,u\}] \text{ and } t \ge 0. \tag{4}$$

Remark 1 Condition (4) states that $g(t,\pm u)$ give, for each t and u, an upper and a lower bound for the oscillations of $g(t,\cdot)$ in the interval $[-u,u]$. A typical situation occurs when

$$l_1|h(t,u)| \le |g(t,u)| \le l_2|h(t,u)| \quad \text{for } (t,u) \in [0,\infty) \times \mathbb{R}$$

for some positive constants l_1 and l_2, and $h(t,u) \in C([0,\infty) \times \mathbb{R})$, with $h(t,\cdot)$ increasing for $t \in [0,\infty)$ and $uh(t,u) > 0$ for $u \ne 0$. Indeed

$$|g(t,v)| \le l_2|h(t,v)| \le l_2|h(t,u)| \le \tfrac{l_2}{l_1}|g(t,u)| \quad \text{for } v \in [\min\{0,u\}, \max\{0,u\}],$$

that is (4) holds with $L = \frac{l_2}{l_1}$. In particular, every g increasing in its second argument satisfies (4) with $L = 1$.

Concerning φ, mainly investigated in previous papers is the case when $\varphi(u) = |u|^{\gamma-1}u$ for some positive α. Under this condition, (4) can be replaced by the weaker assumption (17) simply involving the asymptotic behavior of g. This is possible, in particular, when studying equation (3) where $\alpha = 1$.

2 Unbounded solutions of type I

This section deals with the existence of non-oscillatory type I unbounded solutions of equation (1). A related result on this topic is due to Elbert and Kusano [6] and it treats the case when $g(t,\cdot)$ is increasing in $\mathbb{R}$, for all $t \in [0,\infty)$. Assuming for $k \ne 0$

$$\lim_{\substack{h \to 0 \\ hk > 0}} \frac{\int_0^t \varphi^{-1}\left(\frac{h}{r(s)}\right) ds}{\int_0^t \varphi^{-1}\left(\frac{k}{r(s)}\right) ds} = 0 \tag{5}$$

uniformly in $[t_0,\infty)$ for any $t_0 > 0$, they proved that the existence of constants $k \ne 0$ and $c > 0$ satisfying

$$\int_0^\infty \left| g\left(t, c\int_0^t \varphi^{-1}\left(\frac{k}{r(s)}\right) ds\right) \right| dt < \infty \tag{6}$$

is a necessary and sufficient condition for the appearance of type I solutions. Condition (7) in [6] is indeed slightly different from (5), but one can easily see that they are equivalent. Theorem 1 is a generalization of [6, Theorem 1] since it shows that (6) is a necessary and sufficient condition for the existence of unbounded type I solutions also when g satisfies (4). On this purpose the following lemma is needed, explaining the role of assumption (5) (see also the discussion after the proof of Theorem 1).

Lemma 2 *Assume (4) and (5). Then (6), for some constants $k \neq 0$ and $c > 0$, is equivalent to*

$$\int_0^\infty \left| g\left(t, \int_0^t \varphi^{-1}\left(\frac{h}{r(s)}\right) ds\right)\right| dt < \infty \tag{7}$$

for some $h \neq 0$.

Proof Trivially (7) yields (6) with $c = 1$. On the other hand, if (6) holds for some $k \neq 0$ and $c > 0$, then according to (5), we get the existence of h, with $hk > 0$, $|h| \leq |k|$, and $t_0 > 0$ such that

$$\left|\int_0^t \varphi^{-1}\left(\frac{h}{r(s)}\right) ds\right| \leq c \left|\int_0^t \varphi^{-1}\left(\frac{k}{r(s)}\right) ds\right|$$

for each $t \geq t_0$ and (4) implies (7). □

Theorem 1 *Assume conditions (4) and (5). Then equation (1) has a non-oscillatory solution of type I if and only if (6) holds for some $k \neq 0$ and $c > 0$.*

Proof *Necessary condition.* Let $u(t)$ be a type I solution of equation (1), with

$$\lim_{t\to\infty} r(t)\varphi(u'(t)) = C \neq 0.$$

Take, in particular $C > 0$, implying $u(t)$ eventually positive; with a similar reasoning the case of an eventually negative $u(t)$ can be treated. Hence it is possible to find $\delta > 0$ and $t_0 \geq 0$ such that, for $t \geq t_0$, $u(t) > 0$ and

$$u'(t) > \varphi^{-1}\left(\frac{C-\delta}{r(t)}\right) > 0.$$

Given a sufficiently small $c \in (0, 1]$ such that $u(t_0) \geq c \int_0^{t_0} \varphi^{-1}(\frac{C-\delta}{r(s)})\, ds$, we get, for all $t \geq t_0$,

$$0 \leq c \int_0^t \varphi^{-1}\left(\frac{C-\delta}{r(s)}\right) ds \leq u(t).$$

Then, according to (4), it holds

$$g\left(t, c\int_0^t \varphi^{-1}\left(\frac{C-\delta}{r(s)}\right) ds\right) \leq Lg(t, u(t)),$$

and being

$$\int_0^\infty g(t,u(t))dt = r(0)\varphi(u'(0)) - C,$$

condition (6) holds.

Sufficient condition. Let (6) holds for some constants $k \neq 0$ and $c > 0$. Then, according to Lemma 2, (7) is valid for some $h \neq 0$ with $hk > 0$. With no loss of generality we can assume $k > 0$, so also $h > 0$ and the absolute value in (7) can be removed. Given $l \in (0,h)$, in view of the monotonicity of φ, applying (4) we obtain

$$\int_0^\infty \max_{\int_0^t \varphi^{-1}(\frac{l}{r(s)})ds \le u \le \int_0^t \varphi^{-1}(\frac{h}{r(s)})\,ds} g(t,u)\,dt$$
$$\le L\int_0^\infty g\Big(t, \int_0^t \varphi^{-1}\Big(\frac{h}{r(s)}\Big)\,ds\Big)\,dt < \infty$$

so we can take $t_0 > 0$ such that

$$\int_{t_0}^\infty \max_{\int_0^t \varphi^{-1}(\frac{l}{r(s)})ds \le u \le \int_0^t \varphi^{-1}(\frac{h}{r(s)})\,ds} g(t,u)\,dt \le h - l. \tag{8}$$

Let $C[t_0,\infty)$ be the Fréchet space of all continuous functions $x : [t_0,\infty) \to \mathbb{R}$ with the topology of the uniform convergence on compact subintervals of $[t_0,\infty)$. Let Ω be the closed, convex and bounded subset of $C[t_0,\infty)$ defined as

$$\Omega = \Big\{ w \in C[t_0,\infty) : \int_0^t \varphi^{-1}\Big(\frac{l}{r(s)}\Big)\,ds \le w(t) \le \lambda + \int_{t_0}^t \varphi^{-1}\Big(\frac{h}{r(s)}\Big)\,ds,\ \forall t \ge t_0 \Big\},$$

where $\lambda = \int_0^{t_0} \varphi^{-1}(\frac{l}{r(s)})\,ds$. For every $w \in \Omega$, consider the Cauchy problem

$$\begin{aligned} &(r(t)\varphi(u'))' + g(t,w) = 0,\\ &u(t_0) = \lambda, \quad u'(t_0) = \varphi^{-1}\Big(\tfrac{h}{r(t_0)}\Big). \end{aligned} \tag{9}$$

Since (9) is uniquely solvable, we can define the operator

$$\begin{aligned} T : \Omega &\to C[t_0,\infty)\\ w &\to T(w)(t) = \lambda + \int_{t_0}^t \varphi^{-1}\Big(\frac{h - \int_{t_0}^s g(\eta, w(\eta))d\eta}{r(s)}\Big)\,ds \end{aligned}$$

which associates to any $w \in \Omega$ the unique solution $T(w)$ of problem (9). Now we use the Schauder–Tychonoff fixed point theorem to prove that T has a fixed point. First we show that $T(\Omega) \subseteq \Omega$. In fact, according to the monotonicity of φ and the sign condition (2) on g, one has

$$T(w)(t) \le \lambda + \int_{t_0}^t \varphi^{-1}\Big(\frac{h}{r(s)}\Big)ds \quad \text{for all } t \ge t_0.$$

On the other hand (8) implies

$$T(w)(t) \geq \int_0^t \varphi^{-1}\left(\frac{l}{r(s)}\right) ds \quad \text{for any } t \geq t_0. \tag{10}$$

Now we prove the continuity of T. Let $\{w_n\}$ be a sequence of functions of Ω converging to w, in the topology of $C[t_0, \infty)$, as $n \to \infty$. The continuity of g and φ, the Lebesgue dominated convergence theorem and (8) imply that $T(w_n) \to T(w)$ in $C[t_0, \infty)$ as $n \to \infty$. It remains to prove the relative compactness of T. First notice that $T(\Omega) \subseteq \Omega$, which is bounded in $C[t_0, \infty)$. Moreover

$$(T(w))'(t) = \varphi^{-1}\left(\frac{h - \int_{t_0}^t g(\eta, w(\eta))d\eta}{r(t)}\right);$$

thus, in view of the positivity of g and (8), we get

$$\varphi^{-1}\left(\frac{l}{r(t)}\right) \leq (T(w))'(t) \leq \varphi^{-1}\left(\frac{h}{r(t)}\right) \tag{11}$$

for every $t \geq t_0$ and every $w \in \Omega$. Therefore, the functions in Ω are equicontinuous at each $t \geq t_0$ and Ascoli–Arzelá theorem implies the relative compactness of T. Hence Schauder–Tychonoff theorem can be applied; it guarantees the existence of a function $u \in \Omega$ which remains fixed in T, e.g. of a solution of (1) which is unbounded, in view of (10) and (2). Moreover, from (11) and the monotonicity of φ, u satisfies

$$\lim_{t \to +\infty} r(t)\varphi(u'(t)) = C \in [l, h]. \qquad \square$$

Looking at the proof of Theorem 1, it is clear that (6) is a very natural necessary condition for the existence of type I non-oscillatory solutions of (1). It also follows that (7) is a quite obvious sufficient condition, when employing a fixed point technique for the investigation of type I solutions. As showed in Lemma 2, whenever g satisfies (4) then assumptions (6) and (7) are equivalent, under condition (5). This is the only reason why we introduced (5).

Remark 2 Several results in this framework (see e.g. [5] and [9]) deal with the case when $\varphi(v) = v|v|^{\alpha-1}$ for some $\alpha > 0$. Notice that, for such φ, condition (5) is trivially fulfilled; indeed $\varphi^{-1}(v) = v|v|^{\frac{1}{\alpha}-1}$, hence

$$\lim_{\substack{h \to 0 \\ hk > 0}} \frac{\int_0^t \varphi^{-1}\left(\frac{h}{r(s)}\right) ds}{\int_0^t \varphi^{-1}\left(\frac{k}{r(s)}\right) ds} = \lim_{\substack{h \to 0 \\ hk > 0}} \left(\frac{h}{k}\right)^{\frac{1}{\alpha}} = 0$$

and it is uniform on $[0, \infty)$, for all $k \neq 0$. Moreover it is easy to see that (6) yields (7) with $h = c^\alpha k$. Therefore, (6) and (7) are always equivalent without any additional requirement on g.

Other results (see e.g. [7, 8, 10, 12]) concern the case when $r(t) \equiv 1$. Also under this condition (5) is satisfied, because

$$\lim_{\substack{h\to 0\\ hk>0}} \frac{\int_0^t \varphi^{-1}\big(\frac{h}{r(s)}\big)\,ds}{\int_0^t \varphi^{-1}\big(\frac{k}{r(s)}\big)\,ds} = \lim_{\substack{h\to 0\\ hk>0}} \frac{\varphi^{-1}(h)}{\varphi^{-1}(k)} = 0$$

uniformly on $[0,\infty)$, for all $k \neq 0$.

In the following example we propose a pair of functions $(\varphi(u), r(t))$ which does not satisfy condition (5).

Example 1 Let $\varphi(u) = (\mathrm{e}^{|u|} - 1)\mathrm{sgn}u$ and $r \in C^1[0,\infty)$ such that $r(t) > 0$ for all t and $r(t) \to 0$ as $t \to \infty$. Being $\varphi^{-1}(v) = \log(1 + |v|)\mathrm{sgn}v$, all the assumptions in (2) concerning $\varphi(u)$ and $r(t)$ hold. Moreover, it is easy to see that

$$\lim_{t\to\infty} \frac{\int_0^t \log\big(1 + \frac{|h|}{r(s)}\big)\,ds}{\int_0^t \log\big(1 + \frac{|k|}{r(s)}\big)\,ds} = 1$$

for every choice of h and k with $hk > 0$ and this prevent to condition (5) to be satisfied.

Example 2 Consider the following equation

$$\Big(\frac{u'|u'|^{\alpha-1}}{(1+t)^\beta}\Big)' + q(t)u|u|^{\gamma-1}(a + b\sin^2|u|) = 0, \tag{12}$$

with $\alpha, \gamma, a > 0, \beta \in \mathbb{R}$ and $b \geq 0$. Since, for any $k \neq 0$,

$$\varphi^{-1}\Big(\frac{k}{r(t)}\Big) = k|k|^{\frac{1}{\alpha}-1}(1+t)^{\beta/\alpha}$$

we assume $\beta \geq -\alpha$ for guaranteeing condition (2). In this case, for $(t,u) \in [0,\infty) \times \mathbb{R}$, it holds $aq(t)|u|^\gamma \leq |g(t,u)| \leq (a+b)q(t)|u|^\gamma$. Thus, in view of Remark 1, (4) is satisfied, taking $L = 1 + \frac{b}{a}$. Moreover (6) is equivalent to the convergence of $\int_0^\infty q(t)\big[(1+t)^{\frac{\beta}{\alpha}+1} - 1\big]^\gamma dt$. Therefore, according to Theorem 1, the existence of a non-oscillatory unbounded solution of equation (12) is equivalent to the following condition

$$\int_0^\infty q(t)t^{(\frac{\beta}{\alpha}+1)\gamma}\,dt < \infty. \tag{13}$$

A special case occurs when $\alpha, a = 1$, $\beta, b = 0$ and $q(t) = (1+t)^{-m}$ for some real m. Indeed (12) reduces to the well known generalized Emden–Fowler equation

$$u'' + \frac{1}{(1+t)^m}u|u|^{\gamma-1} = 0. \tag{14}$$

We shall treat again equations (12) and (14) in the end of next sections.

Looking at the proof of Theorem 1, it is easy to deduce that the following stronger sufficient condition (15) is valid, for the existence of a non-oscillatory type I solution. Condition (15) does not require any assumption on φ or g, but it is equivalent to (6) when assuming (4) and (5). The following result holds; we omit its proof, since it is very similar to the sufficient part of Theorem 1.

Proposition 1 *Assume there exists $h < k$ with $hk > 0$ such that*

$$\int_0^\infty \left| \max_{\int_0^t \varphi^{-1}(\frac{h}{r(s)})ds \le u \le \int_0^t \varphi^{-1}(\frac{k}{r(s)})\, ds} g(t,u) \right| dt < \infty. \tag{15}$$

Then equation (1) has a non-oscillatory solution of type I.

Example 3 Consider the equation

$$(r(t)u'(t))' + \frac{\mathrm{e}^{u^2+u^4\sin^2 u}}{(1+t)^2}\operatorname{sign} u = 0, \tag{16}$$

where $r(t)$ satisfies conditions (2) and it is such that $\int_0^t \frac{1}{r(s)}\,ds$ goes to ∞, when $t \to \infty$, as $(\mathrm{loglog}\, t)^\mu$ for some $0 < \mu < 1/4$. Given $t \ge 0$ and an arbitrary value $l \in (0,1)$, it is easy to see that

$$\limsup_{u\to\infty} \frac{g(t,lu)}{g(t,u)} = \limsup_{n\to\infty} \mathrm{e}^{n^2\pi^2(l^2-1+n^2\pi^2 l^4 \sin^2 n\pi l)} = \infty.$$

Therefore condition (4) is not valid and Theorem 1 can not be applied. Take $\beta > 0$ and $T > 0$ satisfying

$$\int_0^t \frac{1}{r(s)}\,ds \le \beta(\log\log\, t)^\mu \quad \text{for all } t \ge T.$$

Given $p \ne 0$, $t \ge T$ and $0 \le u \le |p| \int_0^t \frac{1}{r(s)}\,ds$ it holds

$$0 \le |g(t,u)| \le \frac{\mathrm{e}^{1+2u^4}}{(1+t)^2} \le \frac{\mathrm{e}\,(\log t)^{2\beta^4|p|^4}}{(1+t)^2};$$

this implies (15). According to Proposition 1, also equation (16) has a non-oscillatory solution of type I.

We now consider the special case when φ is a power and prove that not only (5) can be omitted, as showed in Remark 2, but that also (4) can be weakened to an assumption on the asymptotic behavior of g.

Theorem 2 *let $\varphi(v) = v|v|^{\alpha-1}$ for some $\alpha > 0$. Assume the existence of $L \ge 0$ and $m \in (0,1)$ such that*

$$\limsup_{t,|u|\to\infty} \frac{g(t,v)}{g(t,u)} = L \tag{17}$$

for all $v \in [min\{mu,u\}, max\{mu,u\}]$. Then equation (1) has a non-oscillatory solution of type I if and only if (6) holds for some $k \ne 0$ and $c > 0$.

Proof *Necessary condition.* We do not lose in generality when assuming the existence of an eventually positive type I solution of equation (1), i.e. with $\lim_{t\to\infty} r(t)(u'(t))^\alpha = C > 0$. Applying L'Hospital rule we get

$$\lim_{t\to\infty} \frac{u(t)}{\int_0^t \left(\frac{1}{r(s)}\right)^{\frac{1}{\alpha}} ds} = C^{\frac{1}{\alpha}}.$$

Take $\delta > 0$ such that $\frac{C^{\frac{1}{\alpha}}-\delta}{C^{\frac{1}{\alpha}}+\delta} = m$ and $t_0 \geq 0$ satisfying

$$(C^{\frac{1}{\alpha}} - \delta) \int_0^t \left(\frac{1}{r(s)}\right)^{\frac{1}{\alpha}} ds \leq u(t) \leq (C^{\frac{1}{\alpha}} + \delta) \int_0^t \left(\frac{1}{r(s)}\right)^{\frac{1}{\alpha}} ds,$$

for all $t \geq t_0$, that is

$$mu(t) \leq (C^{\frac{1}{\alpha}} - \delta) \int_0^t \left(\frac{1}{r(s)}\right)^{\frac{1}{\alpha}} ds \leq u(t).$$

Then, according to (17), there exists $t_1 \geq t_0$ such that, for $t \geq t_1$, it holds

$$g\Big(t, (C^{\frac{1}{\alpha}} - \delta) \int_0^t \left(\frac{1}{r(s)}\right)^{\frac{1}{\alpha}} ds\Big) \leq 2Lg(t, u(t)),$$

and the conclusion follows as in the proof of Theorem 1.

Sufficient condition. According to Remark 2, (6) implies (7) with $h = c^\alpha k$. For the sake of simplicity let us assume $k > 0$. A similar reasoning holds when $k < 0$. According to (17) and the divergence of $\int_0^\infty (\frac{1}{r(t)})^{\frac{1}{\alpha}} dt$, it is then possible to find $t_0 \geq 0$ such that, for all $t \geq t_0$ and

$$v \in \left[mk^{\frac{1}{\alpha}} \int_0^t \left(\frac{1}{r(s)}\right)^{\frac{1}{\alpha}} ds,\ k^{\frac{1}{\alpha}} \int_0^t \left(\frac{1}{r(s)}\right)^{\frac{1}{\alpha}} ds\right],$$

it holds

$$0 \leq g(t, v) \leq 2Lg\Big(t, k^{\frac{1}{\alpha}} \int_0^t \left(\frac{1}{r(s)}\right)^{\frac{1}{\alpha}} ds\Big).$$

Therefore

$$\int_0^\infty \max_{mk^{\frac{1}{\alpha}} \int_0^t (\frac{1}{r(s)})^{\frac{1}{\alpha}} ds \leq u \leq k^{\frac{1}{\alpha}} \int_0^t (\frac{1}{r(s)})^{\frac{1}{\alpha}} ds} g(t, u)\, dt < \infty,$$

and the conclusion follows from Proposition 1. □

3 Unbounded solutions of type II

We investigate now the existence of type II unbounded solutions $u(t)$ of equation (1), i.e. such that $\lim_{t\to\infty}|u(t)| = \infty$ and $\lim_{t\to\infty} r(t)\varphi(u'(t)) = 0$. Theorem 3 gives a sufficient condition. In the special case (12) we then discuss, in Proposition 2, the existence of a type II solution with prescribed behavior at infinity.

Theorem 3 *Assume condition (4) and let (7) hold for some $h \neq 0$. If*

$$\int_0^\infty \left|\varphi^{-1}\left(\frac{1}{Lr(t)}\int_t^\infty g(s,d)\,ds\right)\right| dt = \infty, \tag{18}$$

for all d satisfying $dh > 0$, then equation (1) has a non-oscillatory solution of type II.

Proof Notice that, with no loss of generality we can assume $h > 0$, implying that also the value d appearing in (18) must be positive. According to (7), it is possible to find $t_0 \geq 0$ such that

$$\int_{t_0}^\infty g\Big(t, \int_0^t \varphi^{-1}\Big(\frac{h}{r(s)}\Big)\,ds\Big)\,dt \leq \frac{h}{L}.$$

Let us denote $d = \int_0^{t_0} \varphi^{-1}\left(\frac{h}{r(s)}\right) ds$. As a consequence of (4) and (7) it follows

$$\int_{t_0}^\infty \max_{d\leq u\leq \int_0^t \varphi^{-1}(\frac{h}{r(s)})\,ds} g(t,u)\,dt \leq L\int_{t_0}^\infty g\left(t, \int_0^t \varphi^{-1}\left(\frac{h}{r(s)}\right) ds\right) dt \leq h. \tag{19}$$

Given the usual Fréchet space of continuous functions $C[t_0,\infty)$, let Ω be its closed, convex and bounded subset defined as follows

$$\Omega = \Big\{w \in C[t_0,+\infty) : d \leq w(t) \leq \int_0^t \varphi^{-1}\left(\frac{h}{r(s)}\right) ds,\ \forall t \geq t_0\Big\}.$$

Since for every $w \in \Omega$, $\int_{t_0}^\infty g(s,w(s))\,ds < \infty$, it is possible to define the operator

$$\begin{aligned} T : \Omega &\to C[t_0,\infty) \\ w &\to T(w)(t) = d + \int_{t_0}^t \varphi^{-1}\left(\frac{\int_s^\infty g(\eta,w(\eta))d\eta}{r(s)}\right) ds \end{aligned}$$

associating to w the unique solution of the Cauchy problem

$$\begin{aligned} &(r(t)\varphi(u'))' + g(t,w) = 0, \\ &u(t_0) = d, \quad u'(t_0) = \varphi^{-1}\left(\frac{\int_{t_0}^\infty g(s,w(s))\,ds}{r(t_0)}\right). \end{aligned} \tag{20}$$

The monotonicity of φ, the sign condition on g and (19) easily yield that $T(\Omega) \subseteq \Omega$. Applying the Schauder–Tychonoff theorem as in the proof of Theorem 1, one can see that T has a fixed element $u(t)$, which is a solution of (1). Moreover, since $u(t) \geq d$ for all $t \geq t_0$, according to (4) and the definition of $T(u)$ it follows

$$u(t) \geq d + \int_{t_0}^{t} \varphi^{-1}\Big(\frac{1}{Lr(s)} \int_{s}^{\infty} g(\eta, d) d\eta\Big) ds;$$

hence condition (18) implies $u(t) \to \infty$ as $t \to \infty$. Finally, since $u(t)$ solves the Cauchy problem (20), it holds

$$r(t)\varphi\big(u'(t)\big) = \int_{t}^{\infty} g\big(s, u(s)\big) ds$$

and by (7) we obtain $r(t)\varphi\big(u'(t)\big) \to 0$ as $t \to \infty$. Consequently $u(t)$ is a type II non-oscillatory solution of equation (1) and the proof is complete. □

Remark 3 In [6, Theorem 3], the case when $g(t, \cdot)$ is increasing for each $t \geq 0$ was studied. Assuming conditions (5), (6) and the natural reformulation of (18) in this context, i.e. with $L = 1$, the authors proved the existence of a type II unbounded solution of equation (1). We recall that condition (5) was introduced only to assure the equivalence between the necessary condition (6) and the sufficient condition (7) (see Lemma 2). However, since we are interested only in the sufficient condition, we don't need any assumption on φ and we directly assumed (7) instead of (6). Therefore, Theorem 3 is a generalization of the quoted result in [6], since it deals with a more general function g and does not require (5). In particular, Theorem 3 holds when $\varphi(u)$ and $r(t)$ behave as in Example 1.

The following part of this section is mainly devoted to equation (12), e.g.

$$\Big(\frac{u'|u'|^{\alpha-1}}{(1+t)^{\beta}}\Big)' + q(t)u|u|^{\gamma-1}(a + b\sin^2|u|) = 0,$$

with $\alpha, \gamma, a > 0$, $\beta \geq -\alpha$ and $b \geq 0$. In this case, condition (18) reduces to

$$\int_{0}^{\infty} \Big(\int_{t}^{\infty} q(s) ds\Big)^{\frac{1}{\alpha}} t^{\frac{\beta}{\alpha}} dt = +\infty. \tag{21}$$

When $q(t) = \frac{1}{(1+t)^m}$, where m is an arbitrary constant, (13) holds if and only if $m > 1 + (1 + \frac{\beta}{\alpha})\gamma$ and (21) is satisfied if and only if $m \leq \alpha + \beta + 1$. Notice that this implies that assumptions (7) and (18) are not always consistent, as follows when $\gamma \geq \alpha$. On the contrary, when $0 < \gamma < \alpha$ and $1+(1+\frac{\beta}{\alpha})\gamma < m \leq 1+\alpha+\beta$ both a type I and a type II unbounded solution exist. When $\alpha, a = 1$, $\beta, b = 0$, (12) reduces to the well known generalized Emden–Fowler equation (14). We recall that its possible solutions of type I are asymptotically linear functions, while the possible solutions of type II are asymptotically sub-linear functions. As a consequence of the analysis above conditions $0 < \gamma < 1$, $1 + \gamma < m \leq 2$ are sufficient for the contemporary presence, in equation (14), of a linear and a sub-linear unbounded solution. We stress that, while condition (6) is necessary for the existence of an unbounded type I solution of (1), neither (7) nor (18) are necessary for the existence of an unbounded type II solution of the same equation. In fact, consider the generalized Emden–Fowler equation with $m = 5/2$ and $\gamma = 2$. Then $\int_0^\infty q(t)t^2 dt = \infty$ and $\int_0^\infty q(t)t\,dt < \infty$ implying that both (7) and (18) are not satisfied; however this equation has the sub-linear solution $u(t) = \frac{\sqrt{t+1}}{4}$.

The following proposition shows that it is possible to determine the exact asymptotic behavior of a type II non-oscillatory solution. In order to simplify notation, we restrict our discussion to equation (12), though a similar investigation could be repeated for the general equation (1).

Proposition 2 *Consider equation (12) with $\alpha, a > 0$, $0 < \gamma < \alpha$, $\beta > -\alpha$, and $b \geq 0$. Given $\sigma \in (0, 1 + \frac{\beta}{\alpha})$, assume that*

$$\int_0^\infty q(t)t^{\sigma\gamma} dt < \infty \tag{22}$$

and

$$t^{1+\frac{\beta}{\alpha}-\sigma}\Big(\int_t^\infty q(s)s^{\sigma\gamma} ds\Big)^{\frac{1}{\alpha}} \to \Delta > 0 \text{ as } t \to \infty. \tag{23}$$

Then equation (12) admits a non-oscillatory solution of type II going at infinity like t^σ when $t \to \infty$.

Proof Let us introduce a continuous function $\vartheta_0 : [0, \infty) \to \mathbb{R}$ satisfying $\vartheta_0(t) = t$ for $t \in [0, 1]$, $\vartheta_0(t) = t^\sigma$ when $t \geq 2$ and $\vartheta_0(t) > 0$ for all $t \neq 0$. According to (22), it holds

$$\int_0^\infty q(t)\vartheta_0^\gamma(t)\,dt < \infty;$$

hence it is possible to define, for $t \geq 0$, the function

$$\psi(t) = \int_0^t (1+s)^{\frac{\beta}{\alpha}}\Big(\int_s^\infty q(\eta)\vartheta_0^\gamma(\eta) d\eta\Big)^{\frac{1}{\alpha}} ds.$$

As a consequence of (23), it follows, as $t \to \infty$

$$t^{1-\sigma}(1+t)^{\frac{\beta}{\alpha}}\Big(\int_s^\infty q(s)\vartheta_0^\gamma(s) ds\Big)^{\frac{1}{\alpha}} \to \Delta,$$

implying $\psi(t) \to \infty$, because $\sigma > 0$, and

$$\lim_{t\to\infty} \frac{\psi(t)}{\vartheta_0(t)} = \lim_{t\to\infty} \frac{t^{1-\sigma}(1+t)^{\frac{\beta}{\alpha}}\Big(\int_t^\infty q(s)s^{\sigma\gamma} ds\Big)^{\frac{1}{\alpha}}}{\sigma} = \frac{\Delta}{\sigma}.$$

Moreover it holds

$$\lim_{t\to 0^+} \frac{\psi(t)}{\vartheta_0(t)} = \Big(\int_0^\infty q(s)\vartheta_0^\gamma(s) ds\Big)^{\frac{1}{\alpha}} > 0.$$

We can then determine two positive constants $0 < m_1 < m_2$ such that $m_1\vartheta_0(t) \leq \psi(t) \leq m_2\vartheta_0(t)$ for all $t \geq 0$. Let

$$d = m_2^{\frac{\alpha}{\alpha-\gamma}}(a+b)^{\frac{1}{\alpha-\gamma}}, \qquad \delta = \frac{a^{\frac{1}{\alpha-\gamma}} m_1^{\frac{\alpha}{\alpha-\gamma}}}{d}$$

and put $\vartheta(t) = d\vartheta_0(t)$. Since $d > 0$ and $0 < \delta < 1$, we can then introduce the closed, convex and bounded set of functions $\Omega = \{w \in C[0,\infty) : \delta\vartheta(t) \leq w(t) \leq \vartheta(t), t \geq 0\}$. According to (22) the operator

$$\begin{aligned} T : \Omega &\to C[t_0, \infty) \\ w &\to T(w)(t) = \int_0^t (1+s)^{\frac{\beta}{\alpha}} \Big(\int_s^\infty q(\eta) w^\gamma(\eta)(a + b\sin^2 w(\eta)) d\eta \Big)^{\frac{1}{\alpha}} ds \end{aligned}$$

is well defined. Now we show that $T(\Omega) \subseteq \Omega$. In fact, given $w \in \Omega$, we have

$$T(w)(t) \leq (a+b)^{\frac{1}{\alpha}} d^{\frac{\gamma}{\alpha}} \psi(t) \leq (a+b)^{\frac{1}{\alpha}} d^{\frac{\gamma}{\alpha}-1} m_2 \vartheta(t).$$

Due to the definition of d it holds $(a+b)^{\frac{1}{\alpha}} d^{\frac{\gamma}{\alpha}-1} m_2 = 1$, implying $T(w)(t) \leq \vartheta(t)$ for all $t \geq 0$. Moreover, since $a^{\frac{1}{\alpha}} d^{\frac{\gamma}{\alpha}-1} m_1 \delta^{\frac{\gamma}{\alpha}-1} = 1$, we get

$$T(w)(t) \geq \delta^{\frac{\gamma}{\alpha}} a^{\frac{1}{\alpha}} d^{\frac{\gamma}{\alpha}} \psi(t) \geq a^{\frac{1}{\alpha}} d^{\frac{\gamma}{\alpha}-1} m_1 \delta^{\frac{\gamma}{\alpha}-1} \delta\vartheta(t) = \delta\vartheta(t).$$

Hence $T(\Omega) \subseteq \Omega$.

As in the proof of Theorem 1, one can apply Schauder–Tychonoff theorem to T in order to show that it has a fixed element $u(t)$; then it is easy to see that $u(t)$ is a solution of equation (12). Finally, according to the definition of the set Ω, $u(t)$ is a type II unbounded solution of (12) satisfying $\frac{u(t)}{t^\sigma} \to l \in [d\delta, d]$ as $t \to \infty$. □

Notice that, since $\sigma \in (0, 1 + \frac{\beta}{\alpha})$, (13) implies (22). Consider again $q(t) = (1+t)^{-m}$. As already showed, equation (12) with $0 < \gamma < \alpha$ and $1 + (1 + \frac{\beta}{\alpha})\gamma < m \leq 1 + \alpha + \beta$ has both a type I and a type II solution. Moreover, take $\sigma = \frac{\alpha+\beta+1-m}{\alpha-\gamma}$. Then $\sigma \in (0, 1 + \frac{\beta}{\alpha})$ and this implies $m - \sigma\gamma > 1$. Therefore, according to Proposition 2, (12) has a type II solution with asymptotic growth t^σ at infinity. In particular, the generalized Emden–Fowler equation (14), with $0 < \gamma < 1$ and $1 + \gamma < m < 2$, contemporarily admits a linear and a sub-linear unbounded solution and the latter one is asymptotic to $t^{\frac{2-m}{1-\gamma}}$.

4 Non-oscillatory theorems

In this section we restrict our attention to equation (3), obtained by (1) when assuming $\varphi(u) = u$. Concerning (3), we state a non-existence result of bounded oscillatory solutions and a non-oscillatory result. Both these problems were extensively investigated and also recent contributions appeared. We refer, in particular, to [3], [5], [10] and [12]. Nevertheless they all treat the cases when $g(t, \cdot)$ is monotone or $g(t, u) = q(t)f(u)$ often assuming $f(u) = |u|^{\gamma-1}u$ for some

$\gamma > 1$. Instead, in Theorems 4 and 5, $g(t,u)$ simply satisfies condition (17), hence no monotonicity is required on it. First notice that now conditions (6) and (7) are equivalent (see Remark 2) and they become

$$\int_0^\infty \left| g\left(t, k\int_0^t \frac{1}{r(s)}\,ds\right)\right| dt < \infty \tag{24}$$

with $k \neq 0$.

Theorem 4 *Assume condition (24) for some $k > 0$ and let (17) hold. Suppose further that for each $v > 0$ there exist $V \geq v$ and $T \geq 0$ satisfying*

$$\sup_{u\in[0,v]} \frac{g(t,u)}{u} \leq \inf_{u\geq V} \frac{g(t,u)}{u} \tag{25}$$

for each $t \in [T,\infty)$. Then equation (3) has no bounded oscillatory solutions.

Proof Let $y(t)$ be an oscillatory solution of (3) and suppose that there exists $t_0 \geq 0$ such that $y(t) \leq 0$ for all $t \geq t_0$. Take $\bar t \geq t_0$ satisfying $y(\bar t) = 0$; then also $y'(\bar t) = 0$ and integrating twice (3) in $[\bar t, t]$, by (2) we obtain

$$y(t) = -\int_{\bar t}^t \frac{1}{r(s)} \int_{\bar t}^s g\big(\sigma, y(\sigma)\big)\,d\sigma\,ds > 0, \quad \text{for all } t > \bar t$$

in contradiction with the sign of $y(t)$. Hence $y(t)$ has positive values for arbitrarily large t. Suppose now that $|y(t)| \leq v$ for some positive v and all $t \geq 0$; let V and T satisfying (25) and take t_1 and t_2, with $T \leq t_1 < t_2$, such that

$$y(t_1) = 0,\ y'(t_2) = 0,\ y'(t) > 0 \quad \text{for all } t_1 \leq t < t_2.$$

According to Theorem 2, we get the existence of an unbounded increasing solution $u(t)$ of (3) satisfying, with no loss of generality, $u(t) \geq V$ in $[t_1, t_2]$. Therefore we obtain, for $t \in [t_1, t_2]$,

$$\begin{aligned}\frac{d}{dt}\Big[r(t)u'(t)y(t) - r(t)y'(t)u(t)\Big] &= \big(r(t)u'(t)\big)'y(t) - \big(r(t)y'(t)\big)'u(t)\\ &= y(t)u(t)\left[\frac{g(t,y(t))}{y(t)} - \frac{g(t,u(t))}{u(t)}\right] \leq 0.\end{aligned}$$

On the other hand,

$$\int_{t_1}^{t_2} \frac{d}{ds}\Big[r(s)u'(s)y(s) - r(s)y'(s)u(s)\Big]ds \geq r(t_2)u'(t_2)y(t_2) + r(t_1)y'(t_1)V > 0,$$

which gives a contradiction. □

Remark 4 Similarly as in the previous theorem, the non-existence of bounded oscillatory solutions for (3) can be obtained when assuming (24) for some $k < 0$, (17) and the condition that for each $v < 0$ there exist $V \leq v$ and $T \geq 0$ such that

$$\sup_{u \in [v,0]} \frac{g(t,u)}{u} \geq \inf_{u \leq V} \frac{g(t,u)}{u}$$

for each $t \in [T, \infty)$.

Remark 5 Cecchi–Marini–Villari [3] obtained the non-existence of bounded oscillatory solutions in the case when $g(t,u) = q(t)f(u)$, assuming, instead of (25), the existence of $\theta \in [0, \infty)$ such that

$$\lim_{u \to 0^+} \frac{f(u)}{u} = \theta \quad \text{and} \quad \lim_{u \to +\infty} \frac{f(u)}{u} = \infty. \tag{26}$$

Notice that, in this case, (25) is equivalent to assume that for each $v > 0$ there exists $V \geq v$ such that

$$\sup_{u \in [0,v]} \frac{f(u)}{u} \leq \inf_{u \in [V,\infty)} \frac{f(u)}{u},$$

which is weaker than (26). In fact, (25) does not require the super-linearity of $\frac{f(u)}{u}$ at infinity, being, for example, fulfilled by any increasing $\frac{f(u)}{u}$.

Under stronger conditions on $r(t)$ and $g(t,u)$, now we give a non-oscillatory result for (3). On this purpose, given a solution $u(t)$ of (3), we introduce the function

$$V_u(t) = \frac{1}{2}\big(r(t)u'(t)\big)^2 + H(t, u(t)), \quad t \geq 0 \tag{27}$$

where

$$H(x,y) = r(x) \int_0^y g(x,s)\, ds, \quad x \geq 0,\, y \in \mathbb{R}.$$

The following estimate is satisfied.

Lemma 3 *Assume that $H_x(x,y)$ exists for $(x,y) \in [0,\infty) \times \mathbb{R}$ and satisfies*

$$H_x(x,y) \leq \rho(x) H(x,y), \quad x \geq 0 \tag{28}$$

where $\rho(t)$ is a non-negative locally integrable function. Then each solution $u(t)$ of (3) satisfies

$$V_u(t) \leq V_u(\tau) e^{\int_\tau^t \rho(s) ds}$$

for all $0 \leq \tau \leq t$.

Proof Given a solution $u(t)$ of (3), consider the function $V_u(t)$ defined in (27). By (28) we get

$$\frac{d}{dt} V_u(t)' \leq \rho(t) H(t, u(t)) \leq \rho(t) V_u(t)$$

for all $t \geq 0$ and the conclusion follows by dividing by $V_u(t)$ and integrating on $[\tau, t]$. □

Remark 6 Notice that when $g(t,u) = q(t)f(u)$ with $q(t) > 0$ and $q(t)r(t)$ absolutely continuous on $[0,\infty)$, then condition (28) holds with

$$\rho(t) = \frac{\big((qr)'(t)\big)_+}{r(t)q(t)} = \frac{\max\{(qr)'(t),0\}}{r(t)q(t)}$$

and previous lemma can be found in [8].

Theorem 5 *Let (24) be satisfied for every $k > 0$. Assume conditions (17) and (28) with*

$$\int_0^\infty \rho(t)dt < \infty. \tag{29}$$

Suppose that there exist $a \geq 1$ and $T \geq 0$ such that (25) is satisfied, for all $v > 0$ and $t \in [T,\infty)$, with $V = av$. Then equation (3) has no oscillatory solutions.

Proof Assume, by contradiction, the existence of an oscillatory solution $y(t)$ of (3) and consider the function $V_y(t)$ defined in (27). According to (29) and Lemma 3, $V_y(t)$ is bounded on all $[0,\infty)$. Hence we get the existence of $k > 0$ such that $|r(t)y'(t)| \leq k$ for $t \geq 0$. As already showed in the proof of Theorem 4, it is possible to prove that $y(t)$ has positive values for arbitrarily large t and to find t_1 and t_2, with $T \leq t_1 \leq t_2$ such that $y(t_1) = 0$, $y'(t_2) = 0$ and $y'(t) > 0$ for all $t_1 \leq t < t_2$. Put $h = \frac{ak}{m}$. According to (24) and reasoning as in the proof of Theorem 2, from (17) we obtain

$$\int_0^\infty \max_{mh\int_0^t \frac{ds}{r(s)} \leq u \leq h\int_0^t \frac{ds}{r(s)}} g(t,u) < \infty.$$

Therefore we can find $t_0 \geq T$ satisfying

$$\int_{t_0}^\infty \max_{mh\int_0^t \frac{ds}{r(s)} \leq u \leq h\int_0^t \frac{ds}{r(s)}} g(t,u) < h(1-m).$$

Notice that it is not restrictive to assume $t_0 \leq t_1$. Reasoning as in Theorem 1, it then follows the existence of a solution $u(t)$ of (3) satisfying

$$u(t) \geq mh\int_{t_1}^t \frac{ds}{r(s)} \geq ay(t) \quad \text{for all } t \in [t_1,t_2].$$

Hence condition (25) can be applied, with $V = av$, implying

$$\frac{g(t,y(t))}{y(t)} - \frac{g(t,u(t))}{u(t)} \leq 0, \quad \text{for } t \in [t_1,t_2].$$

The contradiction then follows when reasoning as in the proof of Theorem 4. □

References

[1] Cecchi, M., Marini, M., Villari, G.: *On some classes of continuable solutions of a nonlinear differential equation.* J. Diff. Equat. **118** (1995), 403–419.

[2] Cecchi, M., Marini, M., Villari, G.: *Topological and variational approaches for nonlinear oscillation: an extension of a Bhatia result.* Proc. First World Congress Nonlinear Analysts, Walter de Gruyter, Berlin, 1996, 1505–1514.

[3] Cecchi, M., Marini, M., Villari, G.: *Comparison results for oscillation of nonlinear differential equations.* Nonlin. Diff. Equat. Appl. **6** (1999), 173–190.

[4] Coffman, C. V., Wong, J. S. W.: *Oscillation and nonoscillation of solutions of generalized Emden–Fowler equations.* Trans. Amer. Math. Soc. **167** (1972), 399–434.

[5] Došlá, Z., Vrkoč, I.: *On an extension of the Fubini theorem and its applications in ODEs.* Nonlinear Anal. **57** (2004), 531–548.

[6] Elbert, A., Kusano, T.: *Oscillation and non-oscillation theorems for a class of second order quasilinear differential equations.* Acta Math. Hung. **56** (1990), 325–336.

[7] Kiyomura, J., Kusano, T., Naito, M.: *Positive solutions of second order quasilinear ordinary differential equations with general nonlinearities.* St. Sc. Math. Hung. **35** (1999), 39–51.

[8] Kusano, T., Norio, Y.: *Nonoscillation theorems for a class of quasilinear differential equations of second order.* J. Math. An. Appl. **189** (1995), 115–127.

[9] Tanigawa, T.: *Existence and asymptotic behaviour of positive solutions of second order quasilinear differential equations.* Adv. Math. Sc. Appl. **9**, 2 (1999), 907–938.

[10] Wang, J.: *On second order quasilinear oscillations.* Funk. Ekv. **41** (1998), 25–54.

[11] Wong, J. S. W.: *On the generalized Emden–Fowler equation.* SIAM Review **17** (1975), 339–360.

[12] Wong, J. S. W.: *A nonoscillation theorem for Emden–Fowler equations.* J. Math. Anal. Appl. **274** (2002), 746–754.

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Infinitesimal Bending of a Subspace of a Space with Non-Symmetric Basic Tensor

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Abstract

In this work infinitesimal bending of a subspace of a generalized Riemannian space (with non-symmetric basic tensor) are studied. Based on non-symmetry of the connection, it is possible to define four kinds of covariant derivative of a tensor. We have obtained derivation formulas of the infinitesimal bending field and integrability conditions of these formulas (equations).

Key words: Generalized Riemannian space, infinitesimal bending, infinitesimal deformation, subspace.

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0 Introduction

0.1. A generalized Riemannian space GR_N is a differentiable manifold, endowed with non-symmetric basic tensor $G_{ij}(x^1,\dots,x^N)$ [2], whose symmetric part is $G_{\underline{ij}}$, and antisymmetric part $G_{\underset{\vee}{ij}}$.

By equations

$$x^i = x^i(u^1,\dots,u^M) \equiv x^i(u^\alpha), \quad \text{rank}(B^i_\alpha) = M, \quad (B^i_\alpha = \partial x^i/\partial u^\alpha), \tag{0.1}$$

in local coordinates is defined a *subspace* $GR_M \subset GR_N$, with metric tensor

$$g_{\alpha\beta} = B^i_\alpha B^j_\beta G_{ij}, \tag{0.2}$$

which is generally also non-symmetric. Remark that in the present work Latin indices $i, j, k, \dots$ take values $1, \dots, N$, while Greek indices $\alpha, \beta, \gamma, \dots$ take values $1, \dots, M$, $(M < N)$ and refer to the subspace.

For the lowering and raising of indices in GR_N one uses the tensor $G_{\underline{ij}}$ respectively $G^{\underline{ij}}$, where $(G^{\underline{ij}}) = (G_{\underline{ij}})^{-1}$.

Christoffel symbols at GR_N are

$$\Gamma_{i.jk} = \frac{1}{2}(G_{ji,k} - G_{jk,i} + G_{ik,j}), \quad \Gamma^i_{jk} = G^{\underline{ip}}\Gamma_{p.jk}, \tag{0.3a, b}$$

where, by the comma a partial derivative is denoted.

The scalar product and the orthogonality one expresses in usual way in the GR_N by $G_{\underline{ij}}$, and in the GR_M by $g_{\underline{\alpha\beta}}$.

On subspaces of generalized Riemannian spaces there exist many works, eg. [7]–[16], [19]–[23]. The present work is continuation and widening of our work [21].

0.2. If in the points of GR_M a vector field $z^i(u^\alpha)$ is defined, the equations

$$\bar{x}^i = x^i(u^\alpha) + \varepsilon z^i(u^\alpha), \tag{0.4}$$

where ε is an infinitesimal, define an *infinitesimal deformation* of the subspace GR_M. Obtained subspace will be denoted $\overline{GR}_M$. The vector field $z^i(u^\alpha)$ is an *infinitesimal deformation field*. In this study of infinitesimal deformations, according to (0.4), magnitudes of a degree higher than the first with respect to ε are omitted.

Among numerous, we refer on papers on infinitesimal deformations of spaces and subspaces, and related topics [4]–[9], [17], [18], [21]–[23].

0.3. A particular case of infinitesimal deformations is *infinitesimal bending* (see e.g. [7], [8], [9], [21]). By virtue of (0.4), for $\overline{g}_{\alpha\beta}$ one obtains [21]:

$$\overline{g}_{\alpha\beta} = g_{\alpha\beta} + \varepsilon(B^i_\alpha B^j_\beta G_{ij,k} z^k + B^i_\alpha z^j_{,\beta} G_{ij} + z^i_{,\alpha} B^j_\beta G_{ij}) \tag{0.5}$$

and, by definition, the subspace $\overline{GR}_M \subset GR_N$ is *infinitesimal bending of the subspace* $GR_M \subset GR_N$ iff (the equation (1.5) in [21]):

$$G_{ij,k} z^k B^i_\alpha B^j_\beta + G_{ij}(B^i_\alpha z^j_{,\beta} + z^i_{,\alpha} B^j_\beta) = 0, \tag{0.6}$$

1 Derivational formulas of the bending field

1.0. Let be $GR_M \subset GR_N$, where GR_M is defined by virtue of (0.1). Consider at points of GR_M $N-M$ mutually orthogonal unit vectors N^i_A, $(A = M+1, \dots, N)$, which are also orthogonal to GR_M, i.e. to the vectors $B^i_\alpha = \partial x^i / \partial u^\alpha$. So, here we are using also the third kind of indices:

$$A, B, C \dots \in \{M+1, \dots, N\}.$$

From the exposed, we have the relations

$$G_{\underline{ip}}G^{\underline{pj}} = \delta_i^j, \quad g_{\underline{\alpha\pi}}g^{\underline{\pi\beta}} = \delta_\alpha^\beta, \tag{1.1a, b}$$

$$G_{\underline{ij}}N^i_{A}B^j_\alpha = 0, \quad G_{\underline{ij}}N^i_{A}N^j_{B} = e_{A}\delta_{AB}, \quad (e_{A} = \pm 1), \tag{1.2a, b}$$

where $g^{\underline{\alpha\beta}}$ is obtained analogously to $G^{\underline{ij}}$. Similarly to (0.3), we can define Cristoffel symbols $\widetilde{\Gamma}^\alpha_{\beta\gamma}$ by means of $g_{\alpha\beta}$. These symbols are in general also non-symmetric. Based on that, for a tensor defined in the points of the subspace we have 4 kinds of covariant derivative. For example [13]:

$$B^i_{\alpha\underset{\substack{1\\2\\3\\4}}{|}\mu} = B^i_{\alpha,\mu} + \Gamma^i_{\substack{pm\\mp\\pm\\mp}}B^p_\alpha B^m_\mu - \widetilde{\Gamma}^\pi_{\substack{\alpha\mu\\\mu\alpha\\\mu\alpha\\\alpha\mu}}B^i_\pi \tag{1.3a–d}$$

$$N^i_{A\underset{\substack{1\\2}}{|}\mu} = N^i_{A\underset{\substack{3\\4}}{|}\mu} = N^i_{A,\mu} + \Gamma^i_{\substack{pm\\mp}}N^p_A B^m_\mu. \tag{1.4a, b}$$

From here one obtains 4 kinds of *derivational formulae* of the subspace $GR_M \subset GR_N$ [13,14]:

$$B^i_{\alpha\underset{\theta}{|}\mu} = \underset{\theta}{\Phi}{}^\pi_{\alpha\mu}B^i_\pi + \sum_{A=M+1}^{N}\Omega_{A\alpha\mu}N^i_{A}, \tag{1.5a}$$

$$N^i_{B\underset{\theta}{|}\mu} = -e_{B}g^{\underline{\pi\sigma}}\underset{\theta}{\Omega}_{B\sigma\mu}B^i_\pi + \sum_{A=M+1}^{N}\underset{\theta}{\Psi}_{AB\mu}N^i_{A}, \underset{\theta}{\Psi}_{BB\mu} = 0, \tag{1.5b}$$

where $\theta \in \{1, 2, 3, 4\}$ designates the kind of covariant derivative. With respect to (4a,b) is:

$$\underset{\substack{1\\2}}{\Omega}_{A\alpha\beta} = \underset{\substack{3\\4}}{\Omega}_{A\alpha\beta} \tag{1.6a, b}$$

$$\underset{\substack{1\\2}}{\Psi}_{AB\mu} = \underset{\substack{3\\4}}{\Psi}_{AB\mu} \tag{1.7a, b}$$

and by virtue of (48′) in [13]:

$$\underset{2}{\Phi}{}^\alpha_{\beta\gamma} = -\underset{1}{\Phi}{}^\alpha_{\beta\gamma}, \quad \underset{3}{\Phi}{}^\alpha_{\beta\gamma} = \underset{1}{\Phi}{}^\alpha_{\beta\gamma} + 2\tilde{\Gamma}^\alpha_{\underset{\vee}{\beta\gamma}},$$
$$\underset{4}{\Phi}{}^\alpha_{\beta\gamma} = -\underset{1}{\Phi}{}^\alpha_{\beta\gamma} - 2\tilde{\Gamma}^\alpha_{\underset{\vee}{\beta\gamma}} \tag{1.8 a,c}$$

1.1. The infinitesimal bending field z^i can be expressed by tangential and normal component with respect to GR_M:

$$z^i = p^\sigma B^i_\sigma + \sum_A q_{A}N^i_{A}. \tag{1.9}$$

Using this value, the condition (0.6) becomes

$$G_{ij,k}B^i_\alpha B^j_\beta(p^\sigma B^k_\sigma+\sum_A q_{A}N^k_{A})$$
$$+g_{\alpha\sigma}p^\sigma_{,\beta}+G_{ij}B^i_\alpha B^j_{\sigma,\beta}p^\sigma+G_{ij}B^i_\alpha\sum_A(q_{A,\beta}N^j_A+q_A N^j_{A,\beta})$$
$$+g_{\sigma\beta}p^\sigma_{,\alpha}+G_{ij}B^j_\beta B^i_{\sigma,\alpha}p^\sigma+G_{ij}B^j_\beta\sum_A(q_{A,\alpha}N^i_A+q_A N^i_{A,\alpha})=0. \quad (1.10)$$

Taking covariant derivative of the kind θ with respect to u^μ and using (5), we get

$$z^i_{\underset{\theta}{|}\mu}=p^\sigma_{\underset{\theta}{|}\mu}B^i_\sigma+p^\sigma B^i_{\sigma\underset{\theta}{|}\mu}+\sum_A(q_{A\underset{\theta}{|}\mu}N^i_A+q_A N^i_{A\underset{\theta}{|}\mu})$$
$$=p^\sigma_{\underset{\theta}{|}\mu}B^i_\sigma+p^\sigma(\underset{\theta}{\Phi}{}^\pi_{\sigma\mu}B^i_\pi+\sum_A\underset{\theta}{\Omega}{}_{A\sigma\mu}N^i_A)+\sum_A q_{A\underset{\theta}{|}\mu}N^i_A$$
$$+\sum_A q_A(-e_A g^{\pi\sigma}\underset{\theta}{\Omega}{}_{A\sigma\mu}B^i_\pi+\sum_B\underset{\theta}{\Psi}{}_{BA\mu}N^i_B),$$

that is

$$z^i_{\underset{\theta}{|}\mu}=\underset{\theta}{P}{}^\pi_\mu B^i_\pi+\sum_A\underset{\theta}{Q}{}_{A\mu}N^i_A, \quad (1.11)$$

where

$$\underset{\theta}{P}{}^\pi_\mu=p^\pi_{\underset{\theta}{|}\mu}+p^\sigma\underset{\theta}{\Phi}{}^\pi_{\sigma\mu}-\sum_A e_A q_A\underset{\theta}{\Omega}{}_{A\sigma\mu}g^{\pi\sigma}, \quad (1.12)$$

$$\underset{\theta}{Q}{}_{A\mu}=p^\sigma\underset{\theta}{\Omega}{}_{A\sigma\mu}+q_{A\underset{\theta}{|}\mu}+\sum_B q_B\underset{\theta}{\Psi}{}_{AB\mu}. \quad (1.13)$$

The equation (11) is *derivational formula of the infinitesimal bending field* z^i. So, we have

Theorem 1.1 *If the infinitesimal bending field z^i of the subspace $GR_M\subset GR_N$ is expressed by the tangential and the normal component with respect to the GR_M in the form (9), then the derivation formula (11) is valid, where $\underset{\theta}{|}\mu$ is covariant derivative of the kind θ according to u^μ, and $\underset{\theta}{P}$, $\underset{\theta}{Q}$ are given in (12) and (13) respectively.*

2 Integrability conditions of derivational formula of the infinitesimal bending field

2.0. Applying to (1.11) covariant derivative of the kind ω with respect to u^ν, we get

$$z^i_{\underset{\theta}{|}\mu\underset{\omega}{|}\nu}=\underset{\theta}{P}{}^\pi_{\mu\underset{\omega}{|}\nu}B^i_\pi+\underset{\theta}{P}{}^\pi_\mu B^i_{\pi\underset{\omega}{|}\nu}+\sum_A(\underset{\theta}{Q}{}_{A\mu\underset{\omega}{|}\nu}N^i_A+\underset{\theta}{Q}{}_{A\mu}N^i_{A\underset{\omega}{|}\nu}),$$

and substituting $B^i_{\pi\underset{\omega}{|}\nu}$ and $N^i_{A\underset{\omega}{|}\nu}$ with respect to (1.5), after arranging one obtains

$$\begin{aligned} z^i_{\underset{\theta}{|}\mu\underset{\omega}{|}\nu} &= [\underset{\theta}{P}{}^{\pi}_{\mu\underset{\omega}{|}\nu} + \underset{\theta}{P}{}^{\sigma}_{\mu}\underset{\omega}{\Phi}{}^{\pi}_{\sigma\nu} - \sum_A e_A \underset{\theta}{Q}{}_{A\mu} g^{\underline{\pi\sigma}} \underset{\omega}{\Omega}{}_{A\sigma\nu}] B^i_\pi \\ &+ \sum_A [\underset{\theta}{P}{}^{\pi}_{\mu}\underset{\omega}{\Omega}{}_{A\pi\nu} + \underset{\theta}{Q}{}_{A\mu\underset{\omega}{|}\nu} + \sum_B \underset{\theta}{Q}{}_{B\mu}\underset{\omega}{\Psi}{}_{AB\nu}] N^i_A, \end{aligned} \tag{2.1}$$

where the tensors $\underset{\theta}{P}$, $\underset{\theta}{Q}$ are given at (1.12,13). From (1) one gets

$$\begin{aligned} z^i_{\underset{\theta}{|}\mu\underset{\omega}{|}\nu} - z^i_{\underset{\omega}{|}\nu\underset{\theta}{|}\mu} &= [\underset{\theta}{P}{}^{\pi}_{\mu\underset{\omega}{|}\nu} - \underset{\omega}{P}{}^{\pi}_{\nu\underset{\theta}{|}\mu} + \underset{\theta}{P}{}^{\sigma}_{\mu}\underset{\omega}{\Phi}{}^{\pi}_{\sigma\nu} - \underset{\omega}{P}{}^{\sigma}_{\nu}\underset{\theta}{\Phi}{}^{\pi}_{\sigma\mu} \\ &- \sum_A e_A g^{\underline{\pi\sigma}} (\underset{\theta}{Q}{}_{A\mu}\underset{\omega}{\Omega}{}_{A\sigma\nu} - \underset{\omega}{Q}{}_{A\nu}\underset{\theta}{\Omega}{}_{A\sigma\mu})] B^i_\pi \\ &+ \sum_A [\underset{\theta}{P}{}^{\pi}_{\mu}\underset{\omega}{\Omega}{}_{A\pi\nu} - \underset{\omega}{P}{}^{\pi}_{\nu}\underset{\theta}{\Omega}{}_{A\pi\mu} + \underset{\theta}{Q}{}_{A\mu\underset{\omega}{|}\nu} - \underset{\omega}{Q}{}_{A\mu\underset{\theta}{|}\mu} \\ &+ \sum_B (\underset{\theta}{Q}{}_{B\mu}\underset{\omega}{\Psi}{}_{AB\nu} - \underset{\omega}{Q}{}_{B\nu}\underset{\theta}{\Psi}{}_{AB\mu})] N^i_A. \end{aligned} \tag{2.2}$$

On the other hand applying the Ricci type identities [11,12], we obtain

$$z^i_{\underset{\substack{1\\2}}{|}\mu\nu} - z^i_{\underset{\substack{1\\2}}{|}\nu\mu} = \underset{\substack{1\\2}}{R}{}^i_{pmn} z^p B^m_\mu B^n_\nu + 2\tilde{\Gamma}^\pi_{\underset{\vee}{\mu\nu}} z^i_{\underset{\substack{1\\2}}{|}\pi}, \tag{2.3a, b}$$

$$z^i_{\underset{1}{|}\mu\underset{2}{|}\nu} - z^i_{\underset{2}{|}\nu\underset{1}{|}\mu} = \underset{3}{R}{}^i_{p\mu\nu} z^p, \tag{2.4}$$

$$z^i_{\underset{\substack{3\\4}}{|}\mu\nu} - z^i_{\underset{\substack{3\\4}}{|}\nu\mu} = \underset{\substack{1\\2}}{R}{}^i_{pmn} z^p B^m_\mu B^n_\nu \pm 2\tilde{\Gamma}^\pi_{\underset{\vee}{\mu\nu}} z^i_{\underset{\substack{1\\2}}{|}\pi}, \tag{2.5a, b}$$

$$z^i_{\underset{3}{|}\mu\underset{4}{|}\nu} - z^i_{\underset{4}{|}\nu\underset{3}{|}\mu} = \underset{4}{R}{}^i_{p\mu\nu} z^p, \tag{2.6}$$

where [11,12]:

$$\underset{1}{R}{}^i_{jmn} = \Gamma^i_{jm,n} - \Gamma^i_{jn,m} + \Gamma^p_{jm}\Gamma^i_{pn} - \Gamma^p_{jn}\Gamma^i_{pm}, \tag{2.7}$$

$$\underset{2}{R}{}^i_{jmn} = \Gamma^i_{mj,n} - \Gamma^i_{nj,m} + \Gamma^p_{mj}\Gamma^i_{np} - \Gamma^p_{nj}\Gamma^i_{mp}, \tag{2.8}$$

$$\begin{aligned} \underset{3}{R}{}^i_{j\mu\nu} &= (\Gamma^i_{jm,n} - \Gamma^i_{nj,m} + \Gamma^p_{jm}\Gamma^i_{np} - \Gamma^p_{nj}\Gamma^i_{pm}) B^m_\mu B^n_\nu \\ &+ 2\Gamma^i_{\underset{\vee}{jm}}(B^m_{\mu,\nu} - \tilde{\Gamma}^\pi_{\nu\mu} B^m_\pi), \end{aligned} \tag{2.9}$$

$$\begin{aligned} \underset{4}{R}{}^i_{j\mu\nu} &= (\Gamma^i_{jm,n} - \Gamma^i_{nj,m} + \Gamma^p_{jm}\Gamma^i_{np} - \Gamma^p_{nj}\Gamma^i_{pm}) B^m_\mu B^n_\nu \\ &+ 2\Gamma^i_{\underset{\vee}{jm}}(B^m_{\mu,\nu} - \tilde{\Gamma}^\pi_{\mu\nu} B^m_\pi). \end{aligned} \tag{2.10}$$

The magnitudes $\underset{1}{R}{}^i_{jmn}$, $\underset{2}{R}{}^i_{jmn}$ are curvature tensors of the first and the second kind respectively of the space GR_N, while the magnitudes $\underset{3}{R}{}^i_{j\mu\nu}$, $\underset{4}{R}{}^i_{j\mu\nu}$ are also tensors and we called them in [11,12] curvature tensors of the space GR_N with respect to the subspace GR_M.

2.1. The cases (3.a,b) can be written in the form

$$z^i_{\underset{\theta}{|}\mu\nu} - z^i_{\underset{\theta}{|}\nu\mu} = \underset{\theta}{R}{}^i_{pmn} z^p B^m_\mu B^n_\nu + 2(-1)^\theta \tilde{\Gamma}^\pi_{\underset{\vee}{\mu\nu}} z^i_{\underset{\theta}{|}\pi}, \theta \in \{1,2\}. \tag{2.11}$$

Taking in (2) $\omega = \theta \in \{1,2\}$, we obtain an equation with the same left side as in (11). Substituting $z^i_{\underset{\theta}{|}\pi}$ in (11) by virtue of (1.11) and equaling the right sides of cited equations, we obtain *the first and the second integrability condition of derivational formula* (1.11) of the infinitesimal bending field z^i of the subspace (for $\theta = 1$, $\theta = 2$):

$$\begin{aligned}
&\underset{\theta}{R}{}^i_{pmn} z^p B^m_\mu B^n_\nu + 2(-1)^\theta \tilde{\Gamma}^\pi_{\underset{\vee}{\mu\nu}} (\underset{\theta}{P}{}^\sigma_\pi B^i_\sigma + \sum_A \underset{\theta}{Q}{}_{A\pi} N^i_A) \\
&\quad = [\underset{\theta}{P}{}^\pi_{\mu\underset{\theta}{|}\nu} - \underset{\theta}{P}{}^\pi_{\nu\underset{\theta}{|}\mu} + \underset{\theta}{P}{}^\sigma_\mu \underset{\theta}{\Phi}{}^\pi_{\sigma\nu} - \underset{\theta}{P}{}^\sigma_\nu \underset{\theta}{\Phi}{}^\pi_{\sigma\mu} \\
&\quad - \sum_A e_A g^{\underline{\pi\sigma}} (\underset{\theta}{Q}{}_{A\mu} \underset{\theta}{\Omega}{}_{A\sigma\nu} - \underset{\theta}{Q}{}_{A\nu} \underset{\theta}{\Omega}{}_{A\sigma\mu})] B^i_\pi \\
&\quad + \sum_A [\underset{\theta}{P}{}^\pi_\mu \underset{\theta}{\Omega}{}_{A\pi\nu} - \underset{\theta}{P}{}^\pi_\nu \underset{\theta}{\Omega}{}_{A\pi\mu} + \underset{\theta}{Q}{}_{A\mu\underset{\theta}{|}\nu} - \underset{\theta}{Q}{}_{A\nu\underset{\theta}{|}\mu} \\
&\quad + \sum_B (\underset{\theta}{Q}{}_{B\mu} \underset{\theta}{\Psi}{}_{AB\nu} - \underset{\theta}{Q}{}_{B\nu} \underset{\theta}{\Psi}{}_{AB\mu})] N^i_A, \quad \theta = 1,2.
\end{aligned} \tag{2.12}$$

a) Multiplying this equation with $G_{\underline{il}} B^l_\lambda$ and using (0.2), (1.1,2), we obtain

$$\begin{aligned}
&\underset{\theta}{R}{}_{lpmn} B^l_\lambda z^p B^m_\mu B^n_\nu + 2(-1)^\theta \tilde{\Gamma}^\pi_{\underset{\vee}{\mu\nu}} \underset{\theta}{P}{}^\sigma_\pi g_{\underline{\lambda\sigma}} \\
&= \Big[\underset{\theta}{P}{}^\pi_{\mu\underset{\theta}{|}\nu} - \underset{\theta}{P}{}^\pi_{\nu\underset{\theta}{|}\mu} + \underset{\theta}{P}{}^\sigma_\mu \underset{\theta}{\Phi}{}^\pi_{\sigma\nu} - \underset{\theta}{P}{}^\sigma_\nu \underset{\theta}{\Phi}{}^\pi_{\sigma\mu} - \sum_A e_A g^{\underline{\pi\sigma}} (\underset{\theta}{Q}{}_{A\mu} \underset{\theta}{\Omega}{}_{A\sigma\nu} - \underset{\theta}{Q}{}_{A\nu} \underset{\theta}{\Omega}{}_{A\sigma\mu})\Big] g_{\underline{\lambda\pi}}.
\end{aligned}$$

Taking into consideration (1.1b) and substituting $\underset{\theta}{P}$, $\underset{\theta}{Q}$ according to (1.12,13), the previous equation becomes

$$
\begin{aligned}
&\underset{\theta}{R}_{lpmn} B^l_\lambda z^p B^m_\mu B^n_\nu + 2(-1)^\theta \tilde\Gamma^\pi_{\underset{\vee}{\mu\nu}} g_{\underline{\lambda\sigma}} (p^\sigma_{\underset{\theta}{|}\pi} + p^\rho \underset{\theta}{\Phi}{}^\sigma_{\rho\pi} - \sum_A e_A q_A \underset{\theta}{\Omega}_{A\rho\pi} g^{\underline{\sigma\rho}})\\
&= [p^\pi_{\underset{\theta}{|}\mu\nu} + p^\sigma_{\underset{\theta}{|}\nu} \underset{\theta}{\Phi}{}^\pi_{\sigma\mu} + p^\sigma \underset{\theta}{\Phi}{}^\pi_{\sigma\mu \underset{\theta}{|}\nu} - \sum_A e_A (q_{A\underset{\theta}{|}\nu} \underset{\theta}{\Omega}_{A\sigma\mu} + q_A \underset{\theta}{\Omega}_{A\sigma\mu\underset{\theta}{|}\nu}) g^{\underline{\pi\sigma}}\\
&- p^\pi_{\underset{\theta}{|}\nu\mu} - p^\sigma_{\underset{\theta}{|}\mu} \underset{\theta}{\Phi}{}^\pi_{\sigma\nu} - p^\sigma \underset{\theta}{\Phi}{}^\pi_{\sigma\nu\underset{\theta}{|}\mu} + \sum_A e_A (q_{A\underset{\theta}{|}\mu} \underset{\theta}{\Omega}_{A\sigma\nu} + q_A \underset{\theta}{\Omega}_{A\sigma\nu\underset{\theta}{|}\mu}) g^{\underline{\pi\sigma}}\\
&+ (p^\sigma_{\underset{\theta}{|}\mu} + p^\rho \underset{\theta}{\Phi}{}^\sigma_{\rho\mu} - \sum_A e_A q_A \underset{\theta}{\Omega}_{A\rho\mu} g^{\underline{\sigma\rho}}) \underset{\theta}{\Phi}{}^\pi_{\sigma\nu}\\
&-(p^\sigma_{\underset{\theta}{|}\nu} + p^\rho \underset{\theta}{\Phi}{}^\sigma_{\rho\nu} - \sum_A e_A q_A \underset{\theta}{\Omega}_{A\rho\nu} g^{\underline{\sigma\rho}}) \underset{\theta}{\Phi}{}^\pi_{\sigma\mu}] g_{\underline{\lambda\pi}}\\
&- \sum_A e_A [(p^\sigma \underset{\theta}{\Omega}_{A\sigma\mu} + q_{A\underset{\theta}{|}\mu} + \sum_B q_B \underset{\theta}{\Psi}_{AB\mu}) \underset{\theta}{\Omega}_{A\lambda\nu}\\
&-(p^\sigma \underset{\theta}{\Omega}_{A\sigma\nu} + q_{A\underset{\theta}{|}\nu} + \sum_B q_B \underset{\theta}{\Psi}_{AB\nu}) \underset{\theta}{\Omega}_{A\lambda\mu}].
\end{aligned}
\tag{2.13}
$$

Substituting the dummy indices l, p with i, j respectively and z^j according to (1.9), using the Ricci type identity

$$
p^\pi_{\underset{\theta}{|}\mu\nu} - p^\pi_{\underset{\theta}{|}\nu\mu} = \underset{\theta}{\tilde R}{}^\pi_{\rho\mu\nu} p^\rho + 2(-1)^\theta \tilde\Gamma^\rho_{\underset{\vee}{\mu\nu}} p^\pi_{\underset{\theta}{|}\rho}, \quad \theta = 1,2 \tag{2.14}
$$

where $\underset{\theta}{\tilde R}{}^\pi_{\rho\mu\nu}$ are the corresponding curvature tensors of the subspace (formed by means of $\tilde\Gamma$) and denoting

$$
p_\lambda = g_{\underline{\lambda\sigma}} p^\sigma, \quad \underset{\theta}{\Phi}_{\lambda\rho\pi} = g_{\underline{\lambda\sigma}} \underset{\theta}{\Phi}{}^\sigma_{\rho\pi},
$$
$$
\underset{\theta}{\Omega}_A{}^\sigma{}_\mu = g^{\underline{\rho\sigma}} \underset{\theta}{\Omega}_{A\rho\mu},
$$

the equation (13) becomes

$$
\begin{aligned}
&\underset{\theta}{R}_{ijmn} B^i_\lambda (p^\sigma B^j_\sigma + \sum_A q_A N^j_A) B^m_\mu B^n_\nu\\
&+2(-1)^\theta \tilde\Gamma^\sigma_{\underset{\vee}{\mu\nu}} (p^\rho \underset{\theta}{\Phi}_{\lambda\rho\sigma} - \sum_A e_A q_A \underset{\theta}{\Omega}_{A\lambda\sigma})\\
&= p^\sigma (\underset{\theta}{\tilde R}_{\lambda\sigma\mu\nu} + \underset{\theta}{\Phi}_{\lambda\sigma\mu\underset{\theta}{|}\nu} - \underset{\theta}{\Phi}_{\lambda\sigma\nu\underset{\theta}{|}\mu} + \underset{\theta}{\Phi}{}^\rho_{\sigma\mu} \underset{\theta}{\Phi}_{\lambda\rho\nu} - \underset{\theta}{\Phi}{}^\rho_{\sigma\nu} \underset{\theta}{\Phi}_{\lambda\rho\mu})\\
&+ \sum_A e_A [q_A (\underset{\theta}{\Phi}_{\lambda\sigma\mu} \underset{\theta}{\Omega}_A{}^\sigma{}_\nu - \underset{\theta}{\Phi}_{\lambda\sigma\nu} \underset{\theta}{\Omega}_A{}^\sigma{}_\mu - \underset{\theta}{\Omega}_{A\lambda\mu\underset{\theta}{|}\nu} + \underset{\theta}{\Omega}_{A\lambda\nu\underset{\theta}{|}\mu})\\
&+p^\sigma (\underset{\theta}{\Omega}_{A\lambda\mu} \underset{\theta}{\Omega}_{A\sigma\nu} - \underset{\theta}{\Omega}_{A\lambda\nu} \underset{\theta}{\Omega}_{A\sigma\mu})\\
&+ \sum_B q_B (\underset{\theta}{\Omega}_{A\lambda\mu} \underset{\theta}{\Psi}_{AB\nu} - \underset{\theta}{\Omega}_{A\lambda\nu} \underset{\theta}{\Psi}_{AB\mu})], \quad \theta = 1,2.
\end{aligned}
\tag{2.15}
$$

b) By multiplying (12) with $G_{\underline{il}}N^l_c$ and taking into consideration (1.1, 2), one obtains

$$\begin{aligned}
&\underset{\theta}{R}_{lpmn}N^l_C z^p B^m_\mu B^n_\nu + 2(-1)^\theta \tilde{\Gamma}^\pi_{\underset{\vee}{\mu\nu}} \underset{\theta}{Q}_{C\pi} e_C \\
&= e_C[\underset{\theta}{P}^\pi_{\ \mu}\underset{\theta}{\Omega}_{C\pi\nu} - \underset{\theta}{P}^\pi_{\ \nu}\underset{\theta}{\Omega}_{C\pi\mu} + \underset{\theta}{Q}_{C\mu\underset{\theta}{|}\nu} - \underset{\theta}{Q}_{C\nu\underset{\theta}{|}\mu} \\
&+ \sum_B(\underset{\theta}{Q}_{B\mu}\underset{\theta}{\Psi}_{CB\nu} - \underset{\theta}{Q}_{B\nu}\underset{\theta}{\Psi}_{CB\mu})].
\end{aligned}$$

Substituting $\underset{\theta}{P}, \underset{\theta}{Q}$ as in the previous case, from here we have

$$\begin{aligned}
&\underset{\theta}{R}_{ijmn}N^i_C z^j B^m_\mu B^N_\nu \\
&+ 2(-1)^\theta \tilde{\Gamma}^\pi_{\underset{\vee}{\mu\nu}} e_C(p^\sigma \underset{\theta}{\Omega}_{C\sigma\pi} + q_{C\underset{\theta}{|}\pi} + \sum_B q_B \underset{\theta}{\Psi}_{CB\pi}) \\
&= e^C\{(p^\pi_{\underset{\theta}{|}\mu} + p^\sigma \underset{\theta}{\Phi}^\pi_{\sigma\mu} - \sum_A e_A q_A \underset{\theta}{\Omega}_{A\sigma\mu} g^{\underline{\pi\sigma}})\underset{\theta}{\Omega}_{C\pi\nu} \\
&- (p^\pi_{\underset{\theta}{|}\nu} + p^\sigma \underset{\theta}{\Phi}^\pi_{\sigma\nu} - \sum_A e_A q_A \underset{\theta}{\Omega}_{A\sigma\nu} g^{\underline{\pi\sigma}})\underset{\theta}{\Omega}_{C\pi\mu} \\
&+ p^\sigma_{\underset{\theta}{|}\nu}\underset{\theta}{\Omega}_{C\sigma\mu} + p^\sigma \underset{\theta}{\Omega}_{C\sigma\mu\underset{\theta}{|}\nu} + q_{C\underset{\theta}{|}\mu\underset{\theta}{|}\nu} \\
&+ \sum_B(q_{B\underset{\theta}{|}\nu}\underset{\theta}{\Psi}_{CB\mu} + q_B \underset{\theta}{\Psi}_{CB\mu\underset{\theta}{|}\nu}) \\
&- p^\sigma_{\underset{\theta}{|}\mu}\underset{\theta}{\Omega}_{C\sigma\nu} - p^\sigma \underset{\theta}{\Omega}_{C\sigma\nu\underset{\theta}{|}\mu} - q_{C\underset{\theta}{|}\nu\underset{\theta}{|}\mu} \\
&- \sum_B(q_{B\underset{\theta}{|}\mu}\underset{\theta}{\Psi}_{CB\nu} + q_B \underset{\theta}{\Psi}_{CB\nu\underset{\theta}{|}\mu}) \\
&+ \sum_B[(p^\sigma \underset{\theta}{\Omega}_{B\sigma\mu} + q_{B\underset{\theta}{|}\mu} + \sum_A q_A \underset{\theta}{\Psi}_{BA\mu})\underset{\theta}{\Psi}_{CB\nu} \\
&- (p^\sigma \underset{\theta}{\Omega}_{B\sigma\nu} + q_{B\underset{\theta}{|}\nu} + \sum_A q_A \underset{\theta}{\Psi}_{BA\nu})\underset{\theta}{\Psi}_{CB\mu}]\}.
\end{aligned}$$

Multiplying the both sides of this equation with $e_C = \pm 1$, and taking into count that

$$\begin{aligned}
q_{C\underset{\theta}{|}\mu} &= \partial q_C/\partial u^\mu = q_{C,\mu}, \quad \theta = 1, 2, \\
q_{C\underset{\frac{1}{2}}{|}\mu\underset{\frac{1}{2}}{|}\nu} &= (q_{C\underset{\frac{1}{2}}{|}\mu})_{,\nu} - \tilde{\Gamma}^\pi_{\underset{\nu\mu}{\mu\nu}} q_{C\underset{\frac{1}{2}}{|}\pi} = q_{C,\mu\nu} - \tilde{\Gamma}^\pi_{\underset{\nu\mu}{\mu\nu}} q_{C,\pi}
\end{aligned}$$

from where $q_{C\underset{\theta}{|}\mu\nu} - q_{C\underset{\theta}{|}\nu\mu} = 2(-1)^\theta \tilde{\Gamma}^\pi_{\underset{\vee}{\nu\mu}} q_{C,\pi}$, the previous equation can be written in the form

$$
\begin{aligned}
& e_C \underset{\theta}{R}_{ijmn} N^i_C \Big(p^\sigma B^j_\sigma + \sum_A q_A N^j_A\Big) B^m_\mu B^n_\nu \\
& +(-1)^\theta \tilde{\Gamma}^\pi_{\underset{\vee}{\mu\nu}} \Big(p^\sigma \underset{\theta}{\Omega}_{C\sigma\pi} + \sum_B q_B \underset{\theta}{\Psi}_{CB\pi}\Big) \\
= {} & p^\sigma \big(\underset{\theta}{\Phi}^\pi_{\sigma\mu} \underset{\theta}{\Omega}_{C\pi\nu} - \underset{\theta}{\Phi}^\pi_{\sigma\nu} \underset{\theta}{\Omega}_{C\pi\mu} + \underset{\theta}{\Omega}_{C\sigma\mu\underset{\theta}{|}\nu} - \underset{\theta}{\Omega}_{C\sigma\nu\underset{\theta}{|}\mu}\big) \\
& + \sum_A e_A q_A \big(\underset{\theta}{\Omega}_{C\pi\mu} \underset{\theta}{\Omega}_{A\nu}{}^{\pi} - \underset{\theta}{\Omega}_{C\pi\nu} \underset{\theta}{\Omega}_{A\mu}{}^{\pi}\big) \\
& + \sum_A \Big[p^\sigma \big(\underset{\theta}{\Omega}_{A\sigma\mu} \underset{\theta}{\Psi}_{CA\nu} - \underset{\theta}{\Omega}_{A\sigma\nu} \underset{\theta}{\Psi}_{CA\mu}\big) \\
& + q_A \big(\underset{\theta}{\Psi}_{CA\mu\underset{\theta}{|}\nu} - \underset{\theta}{\Psi}_{CA\nu\underset{\theta}{|}\mu}\big) \\
& + \sum_B q_B \big(\underset{\theta}{\Psi}_{AB\mu} \underset{\theta}{\Psi}_{CA\nu} - \underset{\theta}{\Psi}_{AB\nu} \underset{\theta}{\Psi}_{CA\mu}\big)\Big].
\end{aligned}
\tag{2.16}
$$

2.2 Substituting $\theta = 1$, $\omega = 2$ into (2) and using (4), we obtain *the third integrability condition* of derivational formula (1.11) of z^i:

$$
\begin{aligned}
\underset{3}{R}^i_{p\mu\nu} z^p = {} & \Big[\underset{1}{P}^\pi_{\mu\underset{2}{|}\nu} - \underset{2}{P}^\pi_{\nu\underset{1}{|}\mu} + \underset{1}{P}^\sigma_\mu \underset{2}{\Phi}^\pi_{\sigma\nu} - \underset{2}{P}^\sigma_\nu \underset{1}{\Phi}^\pi_{\sigma\mu} \\
& - \sum_A e_A g^{\underline{\pi\sigma}} \big(\underset{1}{Q}_{A\mu} \underset{2}{\Omega}_{A\sigma\nu} - \underset{2}{Q}_{A\nu} \underset{1}{\Omega}_{A\sigma\mu}\big)\Big] B^i_\pi \\
& + \sum_A \Big[\underset{1}{P}^\pi_\mu \underset{2}{\Omega}_{A\pi\nu} - \underset{2}{P}^\pi_\nu \underset{1}{\Omega}_{A\pi\mu} + \underset{1}{Q}_{a\mu\underset{2}{|}\nu} - \underset{2}{Q}_{A\nu\underset{1}{|}\mu} \\
& + \sum_B \big(\underset{1}{Q}_{B\mu} \underset{2}{\Psi}_{AB\nu} - \underset{2}{Q}_{B\nu} \underset{1}{\Psi}_{AB\mu}\big)\Big] N^i_A.
\end{aligned}
\tag{2.17}
$$

a) By multiplying the previous equation with $G_{\underline{il}} B^l_\lambda$ one obtains

$$
\begin{aligned}
\underset{3}{R}^i_{lp\mu\nu} B^l_\lambda z^p = {} & \Big[\underset{1}{P}^\pi_{\mu\underset{2}{|}\nu} - \underset{2}{P}^\pi_{\nu\underset{1}{|}\mu} + \underset{1}{P}^\sigma_\mu \underset{2}{\Phi}^\pi_{\sigma\nu} - \underset{2}{P}^\sigma_\nu \underset{1}{\Phi}^\pi_{\sigma\mu} \\
& - \sum_A e_A g^{\underline{\pi\sigma}} \big(\underset{1}{Q}_{A\mu} \underset{2}{\Omega}_{A\sigma\nu} - \underset{2}{Q}_{A\nu} \underset{1}{\Omega}_{A\sigma\mu}\big)\Big] g_{\underline{\lambda\pi}}.
\end{aligned}
$$

By substitution of $\underset{\theta}{P}, \underset{\theta}{Q}$ with respect to (1.12,13), from here it follows that

$$
\begin{aligned}
\underset{3}{R}{}^i_{lp\mu\nu} B^l_\lambda z^p = [& p^\pi_{|\underset{1}{\mu}|\underset{2}{\nu}} + p^\sigma_{|\underset{2}{\nu}} \underset{1}{\Phi}{}^\pi_{\sigma\mu} + p^\sigma \underset{1}{\Phi}{}^\pi_{\sigma\mu|\underset{2}{\nu}} \\
& - \sum_A e_A (q_{A|\underset{2}{\nu}} \underset{1}{\Omega}_{A\sigma\mu} + q_A \underset{1}{\Omega}_{A\sigma\mu|\underset{2}{\nu}}) g^{\underline{\pi\sigma}} \\
& - p^\pi_{|\underset{2}{\nu}|\underset{1}{\mu}} + p^\sigma_{|\underset{1}{\mu}} \underset{2}{\Phi}{}^\pi_{\sigma\nu} \\
& + \sum_A e_A (q_{A|\underset{1}{\mu}} \underset{2}{\Omega}_{A\sigma\nu} + q_A \underset{2}{\Omega}_{A\sigma\nu|\underset{1}{\mu}}) g^{\underline{\pi\sigma}} \\
& + (p^\sigma_{|\underset{1}{\mu}} + p^\sigma \underset{1}{\Phi}{}^\sigma_{\rho\mu} - \sum_A e_A q_A \underset{1}{\Omega}_{A\rho\mu} g^{\underline{\sigma\rho}}) \underset{2}{\Phi}{}^\pi_{\sigma\nu} \\
& - (p^\sigma_{|\underset{2}{\nu}} + p^\sigma \underset{2}{\Phi}{}^\sigma_{\rho\nu} - \sum_A e_A q_A \underset{2}{\Omega}_{A\rho\nu} g^{\underline{\sigma\rho}}) \underset{1}{\Phi}{}^\pi_{\sigma\mu}] g_{\underline{\lambda\pi}} \\
& - \sum_A e_A [(p^\sigma \underset{1}{\Omega}_{A\sigma\mu} + q_{A|\underset{1}{\mu}} + \sum_B q_B \underset{1}{\Psi}_{AB\mu}) \underset{2}{\Omega}_{A\lambda\nu} \\
& - (p^\sigma \underset{2}{\Omega}_{A\sigma\nu} + q_{A|\underset{2}{\nu}} + \sum_B q_B \underset{2}{\Psi}_{AB\nu}) \underset{1}{\Omega}_{A\lambda\mu}].
\end{aligned}
\tag{2.18}
$$

Substituting the dummy indices l, p with i, j respectively and using the Ricci-type identity [11]:

$$
p^\pi_{|\underset{1}{\mu}|\underset{2}{\nu}} - p^\pi_{|\underset{2}{\nu}|\underset{1}{\mu}} = \underset{3}{\tilde{R}}{}^\pi_{\sigma\mu\nu} p^\sigma,
\tag{2.19}
$$

where

$$
\underset{3}{\tilde{R}}{}^\alpha_{\beta\mu\nu} = \tilde{\Gamma}^\alpha_{\beta\mu,\nu} - \tilde{\Gamma}^\alpha_{\nu\beta,\mu} + \tilde{\Gamma}^\sigma_{\beta\mu}\tilde{\Gamma}^\alpha_{\nu\sigma} - \tilde{\Gamma}^\sigma_{\nu\beta}\tilde{\Gamma}^\alpha_{\sigma\mu} + \tilde{\Gamma}^\sigma_{\nu\mu}(\tilde{\Gamma}^\alpha_{\sigma\beta} - \tilde{\Gamma}^\alpha_{\beta\sigma})
\tag{2.20}
$$

is the curvature tensor of the 3^{rd} kind of the subspace, the equation (18) becomes

$$
\begin{aligned}
& \underset{3}{R}_{ij\mu\nu} B^i_\lambda (p^\sigma B^i_\sigma + \sum_A q_A N^i_A) \\
&= p^\sigma (\underset{3}{\tilde{R}}_{\lambda\sigma\mu\nu} + \underset{1}{\Phi}_{\lambda\sigma\mu|\underset{2}{\nu}} - \underset{2}{\Phi}_{\lambda\sigma\nu|\underset{1}{\mu}} + \underset{1}{\Phi}{}^\rho_{\sigma\mu} \underset{2}{\Phi}_{\lambda\rho\nu} - \underset{2}{\Phi}{}^\rho_{\sigma\nu} \underset{1}{\Phi}_{\lambda\rho\mu}) \\
&+ \sum_A e_A [q_A (\underset{1}{\Phi}_{\lambda\sigma\mu} \underset{2}{\Omega}{}^\sigma_{A\nu} - \underset{2}{\Phi}_{\lambda\sigma\nu} \underset{1}{\Omega}{}^\sigma_{A\mu} - \underset{1}{\Omega}_{A\lambda\mu|\underset{2}{\nu}} + \underset{2}{\Omega}_{A\lambda\nu|\underset{1}{\mu}}) \\
&\quad + p^\sigma (\underset{1}{\Omega}_{A\lambda\mu} \underset{2}{\Omega}_{A\sigma\nu} - \underset{2}{\Omega}_{A\lambda\nu} \underset{1}{\Omega}_{A\sigma\mu}) \\
&\quad + \sum_B q_B (\underset{1}{\Omega}_{A\lambda\mu} \underset{2}{\Psi}_{AB\nu} - \underset{2}{\Omega}_{A\lambda\nu} \underset{1}{\Psi}_{AB\mu})].
\end{aligned}
\tag{2.21}
$$

b) Multiplying (17) with $G_{\underline{il}} N^l_c$, one obtains

$$
\begin{aligned}
\underset{3}{R}_{lp\mu\nu} N^l_c z^p = e_c [& \underset{1}{P}{}^\pi_\mu \underset{2}{\Omega}_{C\pi\nu} - \underset{2}{P}{}^\pi_\nu \underset{1}{\Omega}_{C\pi\mu} + \underset{1}{Q}_{c\mu|\underset{2}{\nu}} - \underset{2}{Q}_{c\nu|\underset{1}{\mu}} \\
& + \sum_B (\underset{1}{Q}_{B\mu} \underset{2}{\Psi}_{CB\nu} - \underset{2}{Q}_{B\nu} \underset{1}{\Psi}_{CB\mu})].
\end{aligned}
$$

Substituting $\underset{\theta}{P}, \underset{\theta}{Q}$ using that

$$q_{c|\underset{1}{\mu}|\underset{2}{\nu}} - q_{c|\underset{2}{\nu}|\underset{1}{\mu}} = 0,$$

and arranging, we get

$$\begin{aligned} e_C \underset{3}{R}_{ij\mu\nu} N^i_c \Big(p^\sigma B^j_\sigma + \sum_A q_A N^j_A\Big) &\\ = p^\sigma \big(\underset{1}{\Phi}{}^\pi_{\sigma\mu} \underset{2}{\Omega}_{c\pi\nu} - \underset{2}{\Phi}{}^\pi_{\sigma\nu} \underset{1}{\Omega}_{c\pi\mu} + \underset{1}{\Omega}_{c\sigma\mu|\underset{2}{\nu}} - \underset{2}{\Omega}_{c\sigma\nu|\underset{1}{\mu}}\big) &\\ + \sum_A e_A q_A \big(\underset{1}{\Omega}_{c\pi\mu} \underset{2}{\Omega}{}^\pi_{A\nu} - \underset{2}{\Omega}_{c\pi\nu} \underset{1}{\Omega}{}^\pi_{A\mu}\big) &\\ + \sum_A \Big[p^\sigma \big(\underset{1}{\Omega}_{A\sigma\mu} \underset{2}{\Psi}_{CA\nu} - \underset{2}{\Omega}_{A\sigma\nu} \underset{1}{\Psi}_{CA\mu}\big) + q_A \big(\underset{1}{\Psi}_{CA\mu|\underset{2}{\nu}} - \underset{2}{\Psi}_{CA\nu|\underset{1}{\mu}} &\\ \sum_B q_B \big(\underset{1}{\Psi}_{AB\mu} \underset{2}{\Psi}_{CA\nu} - \underset{2}{\Psi}_{AB\nu} \underset{1}{\Psi}_{CA\mu}\big)\Big]. & \qquad (2.22) \end{aligned}$$

2.3. The cases (5a,b) can be given with the equation

$$\underset{\theta}{z}{}^i_{|\mu\nu} - \underset{\theta}{z}{}^i_{|\nu\mu} = \underset{\theta-2}{R}{}^{\ i}_{pmn} z^p B^m_\mu B^n_\nu + 2(-1)^{\theta-1} \tilde{\Gamma}^\pi_{\underset{\vee}{\mu\nu}} \underset{\theta}{z}{}^i_{|\pi}, \quad \theta \in \{3,4\}. \qquad (2.23)$$

Substituting $\theta \in \{3,4\}$ in (2), we get the equation with the left side as in (21). According to that we get *the 4th and the 5th integrability condition* of the derivation formula (1.11) (for $\theta \in \{3,4\}$):

$$\begin{aligned} \underset{\theta-2}{R}{}^{\ i}_{pmn} z^p B^m_\mu B^n_\nu + 2(-1)^{\theta-1} \tilde{\Gamma}^\pi_{\underset{\vee}{\mu\nu}} \Big(\underset{\theta}{P}{}^\sigma_\pi B^i_\sigma + \sum_A \underset{\theta}{Q}_{A\pi} N^i_A\Big) &\\ = \Big[\underset{\theta}{P}{}^\pi_{\mu|\underset{\theta}{\nu}} - \underset{\theta}{P}{}^\pi_{\nu|\underset{\theta}{\mu}} + \underset{\theta}{P}{}^\sigma_\mu \underset{\theta}{\Phi}{}^\pi_{\sigma\nu} - \underset{\theta}{P}{}^\sigma_\nu \underset{\theta}{\Phi}{}^\pi_{\sigma\mu} &\\ - \sum_A e_A g^{\underline{\pi\sigma}} \big(\underset{\theta}{Q}_{A\mu} \underset{\theta}{\Omega}_{A\sigma\nu} - \underset{\theta}{Q}_{A\nu} \underset{\theta}{\Omega}_{A\sigma\mu}\big)\Big] B^i_\pi &\\ + \sum_A \Big[\underset{\theta}{P}{}^\pi_\mu \underset{\theta}{\Omega}_{A\pi\nu} - \underset{\theta}{P}{}^\pi_\nu \underset{\theta}{\Omega}_{A\pi\mu} + \underset{\theta}{Q}_{A\mu|\underset{\theta}{\nu}} - \underset{\theta}{Q}_{A\nu|\underset{\theta}{\mu}} &\\ + \sum_B \big(\underset{\theta}{Q}_{B\mu} \underset{\theta}{\Psi}_{AB\nu} - \underset{\theta}{Q}_{B\nu} \underset{\theta}{\Psi}_{AB\mu}\big)\Big] N^i_A, \quad \theta \in \{3,4\}. & \qquad (2.24) \end{aligned}$$

a) Multiplying this equation with $G_{\underline{il}} B^l_\lambda$, we get

$$\begin{aligned} \underset{\theta-2}{R}_{lpmn} B^l_\lambda z^p B^m_\mu B^n_\nu + 2(-1)^{\theta-1} \tilde{\Gamma}^\pi_{\underset{\vee}{\mu\nu}} \underset{\theta}{P}{}^\sigma_\pi g_{\underline{\lambda\sigma}} &\\ = \Big[\underset{\theta}{P}{}^\pi_{\mu|\underset{\theta}{\nu}} - \underset{\theta}{P}{}^\pi_{\nu|\underset{\theta}{\mu}} + \underset{\theta}{P}{}^\sigma_\mu \underset{\theta}{\Phi}{}^\pi_{\sigma\nu} - \underset{\theta}{P}{}^\sigma_\nu \underset{\theta}{\Phi}{}^\pi_{\sigma\mu} &\\ - \sum_A e_A g^{\underline{\pi\sigma}} \big(\underset{\theta}{Q}_{A\mu} \underset{\theta}{\Omega}_{A\sigma\nu} - \underset{\theta}{Q}_{A\nu} \underset{\theta}{\Omega}_{A\sigma\mu}\big)\Big] g_{\underline{\lambda\pi}}. & \end{aligned}$$

from where, as in previous cases,

$$
\begin{aligned}
&\underset{\theta-2}{R}{}_{lpmn}B^l_\lambda z^p B^m_\mu B^n_\nu + 2(-1)^{\theta-1}\tilde{\Gamma}^{\pi}_{\underset{\vee}{\mu\nu}} g_{\underline{\lambda\sigma}}(p^\sigma_{\underset{\theta}{|}\pi} + p^\rho \underset{\theta}{\Phi}{}^\sigma_{\rho\pi}\\
&-\sum_A e_A q_A \underset{\theta}{\Omega}{}_{A\rho\pi} g^{\underline{\sigma\rho}}) = [p^\pi_{\underset{\theta}{|}\mu\nu} + p^\sigma_{\underset{\theta}{|}\nu}\underset{\theta}{\Phi}{}^\pi_{\sigma\mu} + p^\sigma \underset{\theta}{\Phi}{}^\pi_{\sigma\mu\underset{\theta}{|}\nu}\\
&-\sum_A e_A(q_{A\underset{\theta}{|}\nu}\underset{\theta}{\Omega}{}_{A\sigma\mu} + q_A\underset{\theta}{\Omega}{}_{A\sigma\mu\underset{\theta}{|}\nu})g^{\underline{\pi\sigma}}\\
&- p^\pi_{\underset{\theta}{|}\nu\mu} - p^\sigma_{\underset{\theta}{|}\mu}\underset{\theta}{\Phi}{}^\pi_{\sigma\nu} - p^\sigma\underset{\theta}{\Phi}{}^\pi_{\sigma\nu\underset{\theta}{|}\mu}\\
&+\sum_A e_A(q_{A\underset{\theta}{|}\mu}\underset{\theta}{\Omega}{}_{A\sigma\nu} + q_A\underset{\theta}{\Omega}{}_{A\sigma\nu\underset{\theta}{|}\mu})g^{\underline{\pi\sigma}}\\
&+(p^\sigma_{\underset{\theta}{|}\mu} + p^\rho\underset{\theta}{\Phi}{}^\sigma_{\rho\mu} - \sum_A e_A q_A\underset{\theta}{\Omega}{}_{A\rho\mu}g^{\underline{\sigma\rho}})\underset{\theta}{\Phi}{}^\pi_{\sigma\nu}\\
&-(p^\sigma_{\underset{\theta}{|}\nu} + p^\rho\underset{\theta}{\Phi}{}^\sigma_{\rho\nu} - \sum_A e_A q_A\underset{\theta}{\Omega}{}_{A\rho\nu}g^{\underline{\sigma\rho}})\underset{\theta}{\Phi}{}^\pi_{\sigma\mu}]g_{\underline{\lambda\pi}}\\
&-\sum_A e_A[(p^\sigma\underset{\theta}{\Omega}{}_{A\sigma\mu} + q_{A\underset{\theta}{|}\mu} + \sum_B q_B\underset{\theta}{\Psi}{}_{AB\mu})\underset{\theta}{\Omega}{}_{A\lambda\nu}]\\
&-(p^\sigma\underset{\theta}{\Omega}{}_{A\sigma\nu} + q_{A\underset{\theta}{|}\nu} + \sum_B q_B\underset{\theta}{\Psi}{}_{AB\nu})\underset{\theta}{\Omega}{}_{A\lambda\mu}].
\end{aligned}
$$

According to [12]:

$$
p^\pi_{\underset{\theta}{|}\mu\nu} - p^\pi_{\underset{\theta}{|}\nu\mu} = \underset{\theta-2}{\tilde{R}}{}^{\pi}{}_{\sigma\mu\nu}p^\sigma + 2(-1)^{\theta-1}\tilde{\Gamma}^{\sigma}_{\underset{\vee}{\mu\nu}} p^\pi_{\underset{\theta}{|}\sigma}, \theta\in\{3,4\}, \tag{2.25}
$$

the previous equation becomes

$$
\begin{aligned}
&\underset{\theta-2}{R}{}_{ijmn}B^i_\lambda(p^\sigma B^j_\sigma + \sum_A q_A N^j_A)B^m_\mu B^n_\nu\\
&+ 2(-1)^{\theta-1}\tilde{\Gamma}^{\pi}_{\underset{\vee}{\mu\nu}}\underset{\theta}{(}p^\sigma\underset{\theta}{\Phi}{}_{\lambda\sigma\pi} - \sum_A e_A q_A\underset{\theta}{\Omega}{}_{A\lambda\pi})\\
&= p^\sigma(\underset{\theta-2}{R}{}_{\lambda\sigma\mu\nu} + \underset{\theta}{\Phi}{}_{\lambda\sigma\mu\underset{\theta}{|}\nu} - \underset{\theta}{\Phi}{}_{\lambda\sigma\nu\underset{\theta}{|}\mu} + \underset{\theta}{\Phi}{}^\rho_{\sigma\mu}\underset{\theta}{\Phi}{}_{\lambda\rho\nu} - \underset{\theta}{\Phi}{}^\rho_{\sigma\nu}\underset{\theta}{\Phi}{}_{\lambda\rho\mu})\\
&+\sum_A e_A[q_A(\underset{\theta}{\Phi}{}_{\lambda\sigma\mu}\underset{\theta}{\Omega}{}^\sigma_{A\nu} - \underset{\theta}{\Phi}{}_{\lambda\sigma\nu}\underset{\theta}{\Omega}{}^\sigma_{A\mu}\underset{\theta}{\Omega}{}_{A\lambda\mu\underset{\theta}{|}\nu} - \underset{\theta}{\Omega}{}_{A\lambda\nu\underset{\theta}{|}\mu})\\
&+p^\sigma(\underset{\theta}{\Omega}{}_{A\lambda\mu}\underset{\theta}{\Omega}{}_{A\sigma\nu} - \underset{\theta}{\Omega}{}_{A\lambda\nu}\underset{\theta}{\Omega}{}_{A\sigma\mu})\\
&+\sum_B q_B(\underset{\theta}{\Omega}{}_{A\lambda\mu}\underset{\theta}{\Psi}{}_{AB\nu} - \underset{\theta}{\Omega}{}_{A\lambda\nu}\underset{\theta}{\Psi}{}_{AB\mu})], \quad \theta\in\{3,4\}
\end{aligned}
\tag{2.26}
$$

b) Multiplying (23) with $G_{\underline{il}}N^l_c$, we have

$$
\begin{aligned}
&\underset{\theta-2}{R}{}_{lpmn}N^l_c z^p B^m_\mu B^n_\nu + 2(-1)^{\theta-1}\tilde{\Gamma}^{\pi}_{\underset{\vee}{\mu\nu}}\underset{\theta}{Q}{}_{c\pi}e_c\\
&= e_c[\underset{\theta}{P}{}^\pi_\mu\underset{\theta}{\Omega}{}_{C\pi\nu} - \underset{\theta}{P}{}^\pi_\nu\underset{\theta}{\Omega}{}_{C\pi\mu} + \underset{\theta}{Q}{}_{c\mu\underset{\theta}{|}\nu} - \underset{\theta}{Q}{}_{c\nu\underset{\theta}{|}\mu}]\\
&+\sum_B(\underset{\theta}{Q}{}_{B\mu}\underset{\theta}{\Psi}{}_{CB\nu} - \underset{\theta}{Q}{}_{B\nu}\underset{\theta}{\Psi}{}_{CB\mu})], \quad \theta\in\{3,4\}.
\end{aligned}
$$

Substituting $\underset{\theta}{P}$, $\underset{\theta}{Q}$, one obtains

$$\begin{aligned}
&e_c \underset{\theta-2}{R}_{ijmn} N^i_c z^j B^m_\mu B^n_\nu \\
&+ 2(-1)^{\theta-1}\tilde{\Gamma}^{\pi}_{\underset{\vee}{\mu\nu}}(p^\sigma \underset{\theta}{\Omega}_{C\sigma\pi} + q_{c\underset{\theta}{|}\pi} + \sum_B q_B \underset{\theta}{\Psi}_{CB\pi}) \\
&= (p^{\pi}_{\underset{\theta}{|}\mu} + p^\sigma \underset{\theta}{\Phi}{}^{\pi}_{\sigma\mu} - \sum_A e_A q_A \underset{\theta}{\Omega}_{A\sigma\mu} g^{\underline{\pi\sigma}}) \underset{\theta}{\Omega}_{C\pi\nu} \\
&-(p^{\pi}_{\underset{\theta}{|}\nu} + p^\sigma \underset{\theta}{\Phi}{}^{\pi}_{\sigma\nu} - \sum_A e_A q_A \underset{\theta}{\Omega}_{A\sigma\nu} g^{\underline{\pi\sigma}}) \underset{\theta}{\Omega}_{C\pi\mu} \\
&+ p^{\sigma}_{\underset{\theta}{|}\nu} \underset{\theta}{\Omega}_{C\sigma\mu} + p^\sigma \underset{\theta}{\Omega}_{C\sigma\mu\underset{\theta}{|}\nu} + q_{C\underset{\theta}{|}\mu\nu} + \sum_B (q_{B\underset{\theta}{|}\nu} \underset{\theta}{\Psi}_{CB\mu} + q_B \underset{\theta}{\Psi}_{CB\underset{\theta}{|}\nu}) \\
&- p^{\sigma}_{\underset{\theta}{|}\mu} \underset{\theta}{\Omega}_{C\sigma\nu} - p^\sigma \underset{\theta}{\Omega}_{C\sigma\nu\underset{\theta}{|}\mu} - q_{C\underset{\theta}{|}\mu\nu} - \sum_B (q_{B\underset{\theta}{|}\nu} \underset{\theta}{\Psi}_{CB\mu} + q_B \underset{\theta}{\Psi}_{CB\underset{\theta}{|}\nu}) \\
&+ \sum_B [(p^\sigma \underset{\theta}{\Omega}_{B\sigma\mu} + q_{B\underset{\theta}{|}\mu} + \sum_A q_A \underset{\theta}{\Psi}_{CB\mu}) \underset{\theta}{\Psi}_{CB\nu} \\
&-(p^\sigma \underset{\theta}{\Omega}_{B\sigma\nu} + q_{B\underset{\theta}{|}\nu} + \sum_A q_A \underset{\theta}{\Psi}_{CB\nu}) \underset{\theta}{\Psi}_{CB\mu}].
\end{aligned}$$

Having in mind that for $\theta \in \{3, 4\}$:

$$q_{C\underset{\theta}{|}\mu\nu} - q_{C\underset{\theta}{|}\mu\nu} = 2(-1)^{\theta-1}\tilde{\Gamma}^{\pi}_{\mu\nu} q_{C,\pi}, \tag{2.27}$$

the previous equation, after putting in order, becomes

$$\begin{aligned}
&e_c \underset{\theta-2}{R}_{ijmn} N^i_c (P^\sigma B^j_\sigma + \sum_A q_A N^j_A) B^m_\mu B^n_\nu \\
&+ 2(-1)^{\theta-1}\tilde{\Gamma}^{\pi}_{\underset{\vee}{\mu\nu}}(p^\sigma \underset{\theta}{\Omega}_{C\sigma\pi} + \sum_B q_B \underset{\theta}{\Psi}{}_{CB\pi}) \\
&= p^\sigma (\underset{\theta}{\Phi}{}^{\pi}_{\sigma\mu} \underset{\theta}{\Omega}_{C\pi\nu} - \underset{\theta}{\Phi}{}^{\pi}_{\sigma\nu} \underset{\theta}{\Omega}_{C\pi\mu} + \underset{\theta}{\Omega}_{C\sigma\mu\underset{\theta}{|}\nu} - \underset{\theta}{\Omega}_{C\sigma\nu\underset{\theta}{|}\mu}) \\
&+ \sum_A e_A q_A (\underset{\theta}{\Omega}{}^{\pi}_{C\pi\mu} \underset{\theta}{\Omega}{}^{\pi}_{A\nu} - \underset{\theta}{\Omega}{}^{\pi}_{C\pi\nu} \underset{\theta}{\Omega}{}^{\pi}_{A\mu}) \\
&+ \sum_A [p^\sigma (\underset{\theta}{\Omega}_{A\sigma\mu} \underset{\theta}{\Psi}_{CA\nu} - \underset{\theta}{\Omega}_{A\sigma\nu} \underset{\theta}{\Psi}_{CA\mu}) + q_A (\underset{\theta}{\Psi}_{CA\mu\underset{\theta}{|}\nu} - \underset{\theta}{\Psi}_{CA\nu\underset{\theta}{|}\mu}) \\
&+ \sum_B q_B (\underset{\theta}{\Psi}_{AB\mu} \underset{\theta}{\Psi}_{CA\nu} - \underset{\theta}{\Psi}_{AB\nu} \underset{\theta}{\Psi}_{CA\mu})], \quad \theta \in \{3, 4\}.
\end{aligned} \tag{2.28}$$

2.4. For $\theta = 3$, $\omega = 4$ according to (2) and (6) we get

$$\begin{aligned}\underset{4}{R}{}^{i}_{p\mu\nu} z^{p} = [&\underset{3}{P}{}^{\pi}_{\mu\underset{4}{|}\nu} - \underset{4}{P}{}^{\pi}_{\nu\underset{3}{|}\mu} + \underset{3}{P}{}^{\sigma}_{\mu}\underset{4}{\Phi}{}^{\pi}_{\sigma\nu} - \underset{4}{P}{}^{\sigma}_{\nu}\underset{3}{\Phi}{}^{\pi}_{\sigma\mu}\\ &\sum_{A} e_{A} g^{\underline{\pi\sigma}}(\underset{3}{Q}_{A\mu}\underset{4}{\Omega}_{A\sigma\nu} - \underset{4}{Q}_{A\nu}\underset{4}{\Omega}_{A\sigma\mu})]B^{i}_{\pi}\\ &+\sum_{A}[\underset{3}{P}{}^{\pi}_{\mu}\underset{4}{\Omega}_{A\pi\nu} - \underset{4}{P}{}^{\pi}_{\nu}\underset{3}{\Omega}_{A\pi\mu} + \underset{3}{Q}_{A\mu\underset{4}{|}\nu} - \underset{4}{Q}_{A\nu\underset{3}{|}\mu}\\ &\quad+\sum_{B}(\underset{3}{Q}_{B\mu}\underset{4}{\Psi}_{AB\nu} - \underset{4}{Q}_{B\nu}\underset{3}{\Psi}_{AB\mu})]N^{i}_{A}.\end{aligned} \tag{2.29}$$

This is the *6th integrability condition* of the derivational formula (1.11) of the deformation field z^i.

a) Multiplying the previous equation with $G_{\underline{il}}B^{l}_{\lambda}$, we get

$$\begin{aligned}\underset{4}{R}_{lp\mu\nu}B^{l}_{\lambda} z^{p} = [&\underset{3}{P}{}^{\pi}_{\mu\underset{4}{|}\nu} - \underset{4}{P}{}^{\pi}_{\nu\underset{3}{|}\mu} + \underset{3}{P}{}^{\sigma}_{\mu}\underset{4}{\Phi}{}^{\pi}_{\sigma\nu} - \underset{4}{P}{}^{\sigma}_{\nu}\underset{3}{\Phi}{}^{\pi}_{\sigma\mu}\\ &\sum_{A} e_{A} g^{\underline{\pi\sigma}}(\underset{3}{Q}_{A\mu}\underset{4}{\Omega}_{A\sigma\nu} - \underset{4}{Q}_{A\nu}\underset{3}{\Omega}_{A\sigma\mu})]g_{\underline{\lambda\pi}}\end{aligned} \tag{2.30}$$

From here, analogously to the previous cases, using the Ricci type identity [12]

$$p^{\pi}\underset{3}{|}\mu\underset{4}{|}\nu - p^{\pi}\underset{4}{|}\nu\underset{3}{|}\mu = \underset{4}{\tilde{R}}{}^{\pi}_{\sigma\mu\nu}p^{\sigma}, \tag{2.31}$$

where

$$\underset{4}{R}{}^{\alpha}_{\beta\mu\nu} = \tilde{\Gamma}^{\alpha}_{\beta\mu,\nu} - \tilde{\Gamma}^{\alpha}_{\nu\beta,\mu} + \tilde{\Gamma}^{\sigma}_{\beta\mu}\tilde{\Gamma}^{\alpha}_{\nu\sigma} - \tilde{\Gamma}^{\sigma}_{\nu\beta}\tilde{\Gamma}^{\alpha}_{\sigma\mu} + \tilde{\Gamma}^{\sigma}_{\mu\nu}(\tilde{\Gamma}^{\alpha}_{\sigma\beta} - \tilde{\Gamma}^{\alpha}_{\beta\sigma}), \tag{2.32}$$

is the 4th kind curvature tensor of a subspace, and from (29) we finally get

$$\begin{aligned}&\underset{4}{R}_{ij\mu\nu}B^{i}_{\lambda}(p^{\sigma}B^{j}_{\sigma} + \sum_{A} q_{A}N^{j}_{A})\\ &= p^{\sigma}(\underset{4}{\tilde{R}}_{\lambda\sigma\mu\nu} + \underset{3}{\Phi}_{\lambda\sigma\mu\underset{4}{|}\nu} + \underset{4}{\Phi}_{\lambda\sigma\nu\underset{3}{|}\mu} + \underset{3}{\Phi}{}^{\rho}_{\sigma\mu}\underset{4}{\Phi}_{\lambda\rho\nu} - \underset{4}{\Phi}{}^{\rho}_{\sigma\nu}\underset{3}{\Phi}_{\lambda\rho\mu})\\ &+\sum_{A} e_{A}[q_{A}(\underset{3}{\Phi}_{\lambda\rho\mu}\underset{4}{\Omega}{}^{\sigma}_{A\nu} - \underset{4}{\Phi}_{\lambda\sigma\nu}\underset{3}{\Omega}{}^{\sigma}_{A\nu} - \underset{3}{\Omega}_{A\lambda\sigma\mu\underset{4}{|}\nu} + \underset{4}{\Omega}_{A\lambda\sigma\nu\underset{3}{|}\mu})\\ &\qquad+p^{\sigma}(\underset{3}{\Omega}_{A\lambda\mu}\underset{4}{\Omega}_{A\sigma\nu} - \underset{4}{\Omega}_{A\lambda\nu}\underset{3}{\Omega}_{A\sigma\mu})\\ &\qquad+\sum_{B} q_{B}(\underset{3}{\Omega}_{A\lambda\mu}\underset{4}{\Psi}_{AB\nu} - \underset{4}{\Omega}_{A\lambda\nu}\underset{3}{\Psi}_{AB\mu})].\end{aligned} \tag{2.33}$$

b) Multiplying (29) with $G_{\underline{il}}N^l_C$ and arranging, we get finally

$$\begin{aligned} & e_C \underset{4}{R}_{ij\mu\nu} N^i_C (p^\sigma B^j_\sigma + \sum_A q_A N^j_A) \\ = & p^\sigma (\underset{3}{\Phi}^\pi_{\sigma\mu} \underset{4}{\Omega}_{C\pi\nu} - \underset{4}{\Phi}^\pi_{\sigma\nu} \underset{3}{\Omega}_{C\pi\mu} + \underset{3}{\Omega}_{C\sigma\mu\underset{4}{|}\nu} - \underset{4}{\Omega}_{C\sigma\nu\underset{3}{|}\mu}) \\ & + \sum_A e_A q_A (\underset{3}{\Omega}_{C\pi\mu} \underset{4}{\Omega}^\pi{}_{A\nu} - \underset{4}{\Omega}_{C\pi\nu} \underset{3}{\Omega}^\pi_{A\mu}) \\ & \sum_A [p^\sigma (\underset{3}{\Omega}_{A\sigma\mu} \underset{4}{\Psi}_{CA\nu} - \underset{4}{\Omega}_{A\sigma\nu} \underset{3}{\Omega}_{CA\mu}) \\ & \quad + q_A (\underset{3}{\Psi}_{CA\mu\underset{4}{|}\nu} - \underset{4}{\Psi}_{CA\nu\underset{3}{|}\mu}) \\ & + \sum_B q_B (\underset{3}{\Psi}_{AB\mu} \underset{4}{\Psi}_{CA\nu} - \underset{4}{\Psi}_{AB\nu} \underset{3}{\Psi}_{CA\mu})] \end{aligned} \tag{2.34}$$

From the above exposed, the next theorem is valid:

Theorem 2.1 *If the infinitesimal bending field z^i of the subspace $GR_M \subset GR_N$ is expressed by virtue of tangent and normal component in the form (1.9), then the coefficients p^σ, q_A satisfy the conditions (15), (16), (21), (22), (26), (28), (33), (34).*

References

[1] Efimov, N. V.: *Kachestvennye voprosy teorii deformacii poverhnostei.* UMN **3.2** (1948), 47–158.

[2] Eisenhart, L. P.: *Generalized Riemann spaces.* Proc. Nat. Acad. Sci. USA **37** (1951), 311–315.

[3] Kon-Fossen, S. E.: Nekotorye voprosy differ. geometrii v celom. *Fizmatgiz, Moskva*, 1959.

[4] Hineva, S. T.: *On infinitesimal deformations of submanifolds of a Riemannian manifold.* Differ. Geom., Banach center publications **12**, *PWN, Warshaw*, 1984, 75–81.

[5] Ivanova-Karatopraklieva, I., Sabitov, I. Kh.: *Surface deformation.* J. Math. Sci., New York, **70**, 2 (1994), 1685–1716.

[6] Ivanova-Karatopraklieva, I., Sabitov, I. Kh.: *Bending of surfaces II.* J. Math. Sci., New York, **74**, 3 (1995), 997–1043.

[7] Lizunova, L. Yu.: *O beskonechno malyh izgibaniyah giperpoverhnostei v rimanovom prostranstve.* Izvestiya VUZ, Matematika **94**, 3 (1970), 36–42.

[8] Markov, P. E.: *Beskonechno malye izgibanya nekotoryh mnogomernyh poverhnostei.* Matemat. zametki **T. 27**, 3 (1980), 469–479.

[9] Mikeš, J.: *Holomorphically projective mappings and their generalizations.* J. Math. Sci., New York, **89**, 3 (1998), 1334–1353.

[10] Mikeš, J., Laitochová, J., Pokorná, O.: *On some relations between curvature and metric tensors in Riemannian spaces.* Suppl. ai Rediconti del Circolo Mathematico di Palermo **2**, 63 (2000), 173–176.

[11] Minčić S. M.: *Ricci type identities in a subspace of a space of non-symmetric affine connexion.* Publ. Inst. Math., NS, t.18(32) (1975), 137–148.

[12] Minčić S. M.: *Novye tozhdestva tipa Ricci v podprostranstve prostranstva nesimmetrichnoi affinoi svyaznosty.* Izvestiya VUZ, Matematika **203**, 4 (1979), 17–27.

[13] Minčić S. M.: *Derivational formulas of a subspace of a generalized Riemannian space.* Publ. Inst. Math., NS, t.34(48) (1983), 125–135.

[14] Minčić S. M.: *Integrability conditions of derivational formulas of a subspace of generalized Riemannian space.* Publ. Inst. Math., NS, t.31(45) (1980), 141–157.

[15] Minčić S. M., Velimirović, L. S.: *O podprostranstvah obob. rimanova prostranstva.* Siberian Mathematical Journal, Dep. v VINITI No.3472-V 98 (1998).

[16] Minčić, S. M., Velimirović, L. S.: *Riemannian Subspaces of Generalized Riemannian Spaces.* Universitatea Din Bacau Studii Si Cercetari Stiintifice, Seria: Matematica, 9 (1999), 111–128.

[17] Minčić, S. M., Velimirović, L. S., Stanković, M. S.: *Infinitesimal Deformations of a Non-Symmetric Affine Connection Space.* Filomat (2001), 175–182.

[18] Velimirović, L. S., Minčić, S. M., Stanković, M. S.: *Infinitesimal deformations and Lie Derivative of a Non-symmetric Affine Connection Space.* Acta Univ. Palacki. Olomuc., Fac. rer. nat., Math. **42** (2003), 111–121.

[19] Mishra, R. S.: *Subspaces of generalized Riemannian space.* Bull. Acad. Roy. Belgique, Cl. sci., (1954), 1058–1071.

[20] Prvanovich, M.: *Équations de Gauss d'un sous-espace plongé dans l'espace Riemannien généralisé.* Bull. Acad. Roy. Belgique, Cl. sci., (1955), 615–621.

[21] Velimirović, L. S., Minčić, S. M.: *On infinitesimal bendings of subspaces of generalized Riemannian spaces.* Tensor (2004), 212–224.

[22] Yano, K.: *Sur la theorie des deformations infinitesimales.* Journal of Fac. of Sci. Univ. of Tokyo **6** (1949), 1–75.

[23] Yano, K.: *Infinitesimal variations of submanifolds.* Kodai Math. J., **1** (1978), 30–44.

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Mathematica **44** (2005) 131–150

Periodic BVP with ϕ-Laplacian and Impulses

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Abstract

The paper deals with the impulsive boundary value problem

$$\frac{d}{dt}[\phi(y'(t))] = f(t, y(t), y'(t)), \quad y(0) = y(T), \quad y'(0) = y'(T),$$
$$y(t_i+) = J_i(y(t_i)), \quad y'(t_i+) = M_i(y'(t_i)), \quad i = 1, \dots m.$$

The method of lower and upper solutions is directly applied to obtain the results for this problems whose right-hand sides either fulfil conditions of the sign type or satisfy one-sided growth conditions.

Key words: ϕ-Laplacian, impulses, lower and upper functions, periodic boundary value problem.

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0 Introduction

In this paper we study the existence of solutions to the following problem

$$\frac{d}{dt}[\phi(y'(t))] = f(t, y(t), y'(t)), \tag{0.1}$$

$$y(0) = y(T), \quad y'(0) = y'(T), \tag{0.2}$$

$$y(t_i+) = J_i(y(t_i)), \quad y'(t_i+) = M_i(y'(t_i)), \quad i = 1, \dots m, \tag{0.3}$$

where $f \in Car([0,T] \times R^2)$, ϕ is an increasing homeomorfismus, $\phi(R) = R$. $J_i \in C(R)$, $M_i \in C(R)$ and

$$y'(t_i) = y'(t_i-) = \lim_{t \to t_i-} y'(t), \quad y'(0) = y'(0+) = \lim_{t \to 0+} y'(t).$$

Let

$$\sigma_1(t_i) < x < \sigma_2(t_i) \Rightarrow J_i(\sigma_1(t_i)) < J_i(x) < J_i(\sigma_2(t_i)), \quad i = 1, \dots, m \tag{0.4}$$

hold. We will assume one of the following properties of M_i, either

$$M_i \text{ is increasing on } R,\ M_i(R) = R \quad i = 1, \dots, m, \tag{0.5}$$

or only

$$\begin{aligned} &y \le \sigma_1'(t_i) \Rightarrow M_i(y) \le M_i(\sigma_1'(t_i)), \\ &y \ge \sigma_2'(t_i) \Rightarrow M_i(y) \ge M_i(\sigma_2'(t_i)), \quad i = 1, \dots, m, \end{aligned} \tag{0.6}$$

In the mathematical literature we can find a lot of papers studying the equation (0.1) with various types of linear or nonlinear boundary conditions. Particularly, the existence results for such problems have been proved e.g. in [1–4].

On the other hand there are papers giving the existence theorems for impulsive problems to the second order differential equations $x'' = f(t, x, x')$. Some of them are based on the method of lower and upper functions ([5–14]). The aim of this paper is to join problems with ϕ-Laplacian and problems with impulses and to extend the method of lower and upper functions for the problem (0.1)–(0.3). Here, the method of lower and upper solutions is directly applied to obtain the results for problems (0.1)–(0.3) whose right-hand sides either fulfil conditions of the sign type or satisfy one-sided growth conditions.

The sections are organized as follows. In Section 1, we begin by definitions of solution and lower and upper functions of the problem (0.1)–(0.3). We state two existence theorems for the problem (0.1)–(0.3) with right-hand sides satisfying conditions of the sign type and one-sided growth conditions and show some applications of these theorems on the concrete problems. In Section 2, we state and prove the existence result for problems with bounded right-hand sides. This problem is reduced to a fixed point problem and using the Schauder fixed point theorem, we show its solvability. In Section 3, we use the previous result to prove the existence theorems which are stated in Section 1.

1 Formulation of the solution and main results

For a real valued function u defined a.e. on $[0,T]$, we put

$$\|u\|_\infty = \sup_{t \in [0,T]} \operatorname{ess} |u(t)|.$$

Let $m \in N$ and $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = T$ be a division of the interval $J = [0,T]$. We denote $\Delta = \{t_1, t_2, \dots, t_m\}$ and define $C^1_\Delta(J)$, resp. $C_\Delta(J)$, as the set of functions $u : J \to R$,

$$u(t) = \begin{cases} u^{[0]}(t), & t \in [0, t_1], \\ u^{[1]}(t), & t \in (t_1, t_2], \\ \dots & \dots \\ u^{[m]}(t), & t \in (t_m, T], \end{cases}$$

where $u^{[i]} \in C^1[t_i, t_{i+1}]$, resp. $u^{[i]} \in C[t_i, t_{i+1}]$, for $i = 0, 1, \dots, m$. Moreover, $AC_\Delta(J)$ stands for the set of functions $u \in C_\Delta(J)$ being absolutely continuous on each subinterval (t_i, t_{i+1}), $i = 0, 1, \dots, m$. For $u \in C^1_\Delta(J)$ we write

$$\|u\|_{C^1_\Delta(J)} = \|u\|_\infty + \|u'\|_\infty.$$

Definition 1 A *solution of the problem* (0.1)–(0.3) is a function $y \in C^1_\Delta(J)$ such that $\phi(y') \in AC_\Delta(J)$, y fulfils equation (0.1) for a.e. $t \in J$, further satisfies the periodic conditions (0.2) and the impulsive conditions (0.3).

Definition 2 Functions $\sigma_1 \in C^1_\Delta(J)$, $\sigma_2 \in C^1_\Delta(J)$ are respectively called *lower and upper functions of the problem* (0.1)–(0.3), if $\phi(\sigma_1'), \phi(\sigma_2') \in AC_\Delta(J)$ and

$$(\phi(\sigma_1'(t)))' \geq f(t, \sigma_1(t), \sigma_1'(t)), \quad (\phi(\sigma_2'(t)))' \leq f(t, \sigma_2(t), \sigma_2'(t)) \quad \text{for a.e. } t \in J,$$
$$\sigma_1(0) = \sigma_1(T), \quad \sigma_2(0) = \sigma_2(T),$$
$$\sigma_1'(0) \geq \sigma_1'(T), \quad \sigma_2'(0) \leq \sigma_2'(T),$$
$$\sigma_1(t_i+) = J_i(\sigma_1(t_i)), \quad \sigma_2(t_i+) = J_i(\sigma_2(t_i)), \quad i = 1, \dots, m,$$
$$\sigma_1'(t_i+) \geq M_i(\sigma_1'(t_i)), \quad \sigma_2'(t_i+) \leq M_i(\sigma_2'(t_i)), \quad i = 1, \dots, m.$$

Remark 1.1 If $M_i(0) = 0$ for $i = 1, \dots, m$ and $r_1 \in R$ is such that $J_i(r_1) = r_1$ for $i = 1, \dots, m$ and

$$f(t, r_1, 0) \leq 0 \quad \text{for a.e. } t \in J,$$

then $\sigma_1(t) \equiv r_1$ on J is a lower function of the problem (0.1)–(0.3). Similarly, if $r_2 \in R$ is such that $J_i(r_2) = r_2$ for $i = 1, \dots, m$ and

$$f(t, r_2, 0) \geq 0 \quad \text{for a.e.} t \in J,$$

then $\sigma_2(t) \equiv r_2$ on J is an upper function of the problem (0.1)–(0.3).

The main results of this paper are contained in the following two theorems. In Theorem 1.1 we suppose that the right-hand side f of equation (0.1) fulfils conditions of the sign type.

Theorem 1.1 *Let lower and upper functions of the problem (0.1)–(0.3) exist and satisfy (0.4), (0.6) and $\sigma_1 \leq \sigma_2$ on J. Let there exist functions $\varphi_1, \varphi_2 \in C_\Delta(J)$ such that $\phi(\varphi_1), \phi(\varphi_2) \in AC_\Delta(J)$ and*

$$\begin{aligned} &\varphi_1(0) \geq \varphi_1(T), \quad \varphi_2(0) \leq \varphi_2(T), \\ &\varphi_1(t) \leq \sigma_i'(t) \leq \varphi_2(t), \text{ on } J, \ i = 1, 2, \\ &\varphi_1(t_j+) \geq M_j(\varphi_1(t_j)), \quad \varphi_2(t_j+) \leq M_j(\varphi_2(t_j)), \quad j = 1, \dots, m. \end{aligned} \tag{1.7}$$

Furthermore, let φ_1, φ_2 satisfy inequalities

$$f(t,x,\varphi_1(t)) \le (\phi(\varphi_1(t)))', \quad f(t,x,\varphi_2(t)) \ge (\phi(\varphi_2(t)))' \tag{1.8}$$

for a.e. $t \in J$ and for all $x \in [\sigma_1(t), \sigma_2(t)]$.

Then the problem (0.1)–(0.3) has a solution $u \in C^1_\Delta(J)$ such that

$$\sigma_1 \le u \le \sigma_2, \quad \varphi_1 \le u' \le \varphi_2 \quad \textit{on } J. \tag{1.9}$$

Remark 1.2 If $s_1 \le \sigma_j'(t)$ on J, $j = 1,2$, is such that $M_i(s_1) = s_1$ for $i = 1, \dots, m$ and

$$f(t,x,s_1) \le 0 \quad \text{for a.e. } t \in J, \text{ for all } x \in [\sigma_1(t), \sigma_2(t)],$$

then $\varphi_1(t) \equiv s_1$ on J fulfils conditions of Theorem 1.1. If $s_2 \ge \sigma_j'(t)$ on J, $j = 1,2$, is such that $M_i(s_2) = s_2$ for $i = 1, \dots, m$ and

$$f(t,x,s_2) \ge 0 \quad \text{for a.e. } t \in J, \text{ for all } x \in [\sigma_1(t), \sigma_2(t)],$$

then $\varphi_2(t) \equiv s_2$ on J fulfils conditions of Theorem 1.1.

Example 1.1

$$\begin{aligned}
&\tfrac{d}{dt}[\phi(x')] = t^p + x^q + (x')^r + \tfrac{\sqrt{T}}{\sqrt{t}}(x')^k, \quad x(0) = x(T), \quad x'(0) = x'(T),\\
&x(t_i+) = a_i(x(t_i))^2 + (1 - a_i(A+B))x(t_i) + ABa_i = J_i(x(t_i)),\\
&\qquad i = 1, \dots, m,\\
&x'(t_i+) = b_i(x'(t_i))^3 - b_i(D+C)(x'(t_i))^2 + (1 + b_iCD)x'(t_i) = M_i(x'(t_i)),\\
&\qquad i = 1, \dots, m,
\end{aligned} \tag{1.10}$$

$k > 0$ and $q > 0$ are odd, $p > 0$, $r > 0$, $A < 0$, $B > 0$, $C < 0$, $D > 0$. If $a_i \in [-\frac{1}{B-A}, \frac{1}{B-A}]$, $i = 1, \dots, m$, then J_i satisfy condition (0.4) for $i = 1, \dots, m$. If $b_i \in [0, \frac{4}{(D-C)^2}]$, $i = 1, \dots, m$, then M_i satisfy condition (0.6) for $i = 1, \dots, m$. $J_i(A) = A, J_i(B) = B, M_i(C) = C, M_i(D) = D$, $i = 1, \dots, m$.

If $A^q + T^p \le 0$ then $\sigma_1(t) \equiv A$ is a lower function of the problem (1.10). Function $\sigma_2(t) \equiv B$ is an upper function of the problem (1.10). Further, if $B^q + T^p \le -C^k - C^r$ and $|A|^q \le D^k + D^r$, then functions $\varphi_1(t) \equiv C$, $\varphi_2(t) \equiv D$ satisfy the conditions of Theorem 1.1, so there exists a solutions of the problem (1.10) fulfiling inequalities (1.9).

Example 1.2

$$\begin{aligned}
&\left((x')^3\right)' = \tfrac{1}{\sqrt{t}}(x'^k - \operatorname{sgn} x') + x^p + t^q, \quad k > 0,\ p > 0 \text{ are odd},\ q \ge 0,\\
&x(0) = x(3), \quad x'(0) = x'(3),\\
&x(1+) = x(1) + 1, \quad x'(1+) = x'(1) - 2,\\
&x(2+) = x(2) - 2, \quad x'(2+) = x'(2) + 2.
\end{aligned} \tag{1.11}$$

If we select functions σ_1 and σ_2 in the following way

$$\sigma_1 = \begin{cases} t+1-4\cdot 3^{\frac{q}{p}}, & t\in[0,1],\\ -t+4-4\cdot 3^{\frac{q}{p}}, & t\in(1,2],\\ t-2-4\cdot 3^{\frac{q}{p}}, & t\in(2,3], \end{cases}$$

$$\sigma_2 = \begin{cases} t-2+6\cdot 3^{\frac{q}{p}}, & t\in[0,1],\\ -t+1+6\cdot 3^{\frac{q}{p}}, & t\in(1,2],\\ t-5+6\cdot 3^{\frac{q}{p}}, & t\in(2,3], \end{cases}$$

then σ_1, σ_2 are respectively lower and upper functions of the problem (1.11). If we select functions φ_1 and φ_2 in this way

$$\varphi_1 = \begin{cases} -6^{\frac{p+1}{k}}\cdot 3^{\frac{q}{pk}}, & t\in[0,1],\\ -6^{\frac{p+1}{k}}\cdot 3^{\frac{q}{pk}}-2, & t\in(1,2],\\ -6^{\frac{p+1}{k}}\cdot 3^{\frac{q}{pk}}, & t\in(2,3], \end{cases}$$

$$\varphi_2 = \begin{cases} 4^{\frac{p+1}{k}}\cdot 3^{\frac{q}{pk}}+2, & t\in[0,1],\\ 4^{\frac{p+1}{k}}\cdot 3^{\frac{q}{pk}}, & t\in(1,2],\\ 4^{\frac{p+1}{k}}\cdot 3^{\frac{q}{pk}}+2, & t\in(2,3], \end{cases}$$

then these functions satisfy the conditions of Theorem 1.1, so there exists a solutions of the problem (1.11) fulfiling inequalities (1.9).

In Theorem 1.2 we impose one-sided conditions of the growth type on f.

Theorem 1.2 *Let σ_1, σ_2 be respectively lower and upper functions of the problem (0.1)–(0.3) and satisfy (0.4), (0.5) and $\sigma_1 \le \sigma_2$ on J. Assume that $k \in L(J)$ is nonnegative a.e. on $[0,T]$, $\omega \in C([0,\infty))$ is positive on $[0,\infty)$ and*

$$\int_{-\infty}^{\phi(-1)} \frac{ds}{\omega(|\phi^{-1}(s)|)} = \infty, \qquad \int_{\phi(1)}^{\infty} \frac{ds}{\omega(|\phi^{-1}(s)|)} = \infty$$

and

$$f(t,x,y) \le \omega(|y|)(k(t)+|y|) \text{ for a.e. } t\in J \text{ and every } (x,y)\in[\sigma_1(t),\sigma_2(t)]\times R. \tag{1.12}$$

Then the problem (0.1)–(0.3) has a solution u such that $\sigma_1 \le u \le \sigma_2$ on J.

Example 1.3

$$\begin{aligned} &\left(|x'|^{k-1}x'\right)' = \tfrac{1}{\sqrt{t}}(x'^k-1)+x^m+x'^{k+1}, \quad k>0 \text{ even},\ m>0 \text{ odd},\\ &x(0)=x(3), \quad x'(0)=x'(3),\\ &x(1+)=x(1)+1, \quad x'(1+)=x'(1)-2,\\ &x(2+)=x(2)-2, \quad x'(2+)=x'(2)+2. \end{aligned} \tag{1.13}$$

Define functions $\sigma_i : J \to R$, $i = 1, 2$

$$\sigma_1(t) = \begin{cases} t-3 & \text{if } \ t \in [0,1], \\ -t & \text{if } \ t \in [1,2], \\ t-6 & \text{if } \ t \in [2,3], \end{cases} \qquad \sigma_2(t) = \begin{cases} t+1 & \text{if } \ t \in [0,1], \\ -t+4 & \text{if } \ t \in [1,2], \\ t-2 & \text{if } \ t \in [2,3]. \end{cases}$$

Then we have

$$\begin{aligned} f(t, \sigma_1, \sigma_1') &= \frac{1}{\sqrt{t}}\big(\sigma_1'^2 - 1\big) + \sigma_1^3 + \sigma_1'^3 \\ &= \left\{ \begin{array}{ll} \frac{1}{\sqrt{t}}(1-1) + (t-3)^m + 1 < 0 & \text{if } t \in [0,1] \\ \frac{1}{\sqrt{t}}(1-1) + (-t)^m - 1 < 0 & \text{if } t \in (1,2] \\ \frac{1}{\sqrt{t}}(1-1) + (t-6)^m + 1 < 0 & \text{if } t \in (2,3] \end{array} \right\} = \big(\phi(\sigma_1')\big)', \end{aligned}$$

$$\begin{aligned} f(t, \sigma_2, \sigma_2') &= \frac{1}{\sqrt{t}}\big(\sigma_2'^2 - 1\big) + \sigma_2^3 + \sigma_1'^3 \\ &= \left\{ \begin{array}{ll} \frac{1}{\sqrt{t}}(1-1) + (t+1)^m + 1 > 0 & \text{if } t \in [0,1] \\ \frac{1}{\sqrt{t}}(1-1) + (-t+4)^m - 1 > 0 & \text{if } t \in (1,2] \\ \frac{1}{\sqrt{t}}(1-1) + (t-2)^m + 1 > 0 & \text{if } t \in (2,3] \end{array} \right\} = \big(\phi(\sigma_2')\big)'. \end{aligned}$$

Functions σ_1, σ_2 are respectively lower and upper functions of the problem (1.13). The right-hand side of the equation does not fulfil conditions of the sign type, because $f(t, x, \varphi_1)$ is not bounded on $[0, 1]$. Nevertheless, one-sided conditions of the growth type are valid.

$$\phi^{-1}(x) = |x|^{\frac{1}{k}} \operatorname{sgn} x, \qquad \omega(s) = 1 + s^k,$$

$$\int_1^{\infty} \frac{ds}{\omega(|\phi^{-1}(s)|)} = \infty, \qquad \int_{-\infty}^{-1} \frac{ds}{\omega(|\phi^{-1}(s)|)} = \infty,$$

$$\begin{aligned} f(t,x,y) \ &= \ \frac{1}{\sqrt{t}}\big(y^k - 1\big) + x^m + y^{k+1} \le \frac{1}{\sqrt{t}}(|y|^k + 1) + (\sigma_2^m(t) + |y|)(|y|^k + 1) \\ &\le \ (1 + |y|^k)(\frac{1}{\sqrt{t}} + \sigma_2^m(t) + |y|) = \omega(|y|)(k(t) + |y|). \end{aligned}$$

By means of Theorem 1.2, there exists a solution of the problem (1.13).

2 Existence result for bounded right-hand sides of equations

At the beginning of this section we introduce an auxiliary problem and find a priori estimates for its solution. The main result of this section is contained in Theorem 2.1. In the proof of this theorem we show that a solution of the auxiliary problem (2.6)–(2.9) exists and is also a solution of the problem (0.1)–(0.3).

Assume that there is $h \in L(J)$ such that $|f(t,x,y)| \le h(t)$ for a.e. $t \in J$, for all $(x,y) \in [\sigma_1(t), \sigma_2(t)] \times R$. Define function $\varphi : J \times R \to R$

$$\varphi(t,x) = \begin{cases} \sigma_2(t) & \text{if } x > \sigma_2(t), \\ x & \text{if } \sigma_1(t) \le x \le \sigma_2(t), \\ \sigma_1(t) & \text{if } x < \sigma_1(t), \end{cases} \tag{2.1}$$

and further functions $\omega_i : J \times [0,1] \to R,\ i = 1,2,$

$$\begin{aligned} \omega_1(t,\varepsilon) &= \sup\{|f(t,\sigma_1,\sigma_1') - f(t,\sigma_1,y)| : |y - \sigma_1'| \le \varepsilon\}, \\ \omega_2(t,\varepsilon) &= \sup\{|f(t,\sigma_2,\sigma_2') - f(t,\sigma_2,y)| : |y - \sigma_2'| \le \varepsilon\}. \end{aligned} \tag{2.2}$$

We see that $\omega_i \in Car(J \times [0,1])$ are nonnegative, nondecreasing in the second variable and $\omega_i(t,0) = 0$ for a.e. $t \in J$, $i = 1,2$.

Now, define $F : J \times R^2 \to R$ such that

$$F(t,x,y) = \begin{cases} f(t,\sigma_2,y) + \omega_2(t, \frac{x-\sigma_2}{x-\sigma_2+1}) + \frac{x-\sigma_2}{x-\sigma_2+1} & \text{for } x > \sigma_2(t), \\ f(t,x,y) & \text{for } \sigma_1(t) \le x \le \sigma_2(t), \\ f(t,\sigma_1,y) - \omega_1(t, \frac{\sigma_1-x}{\sigma_1-x+1}) - \frac{\sigma_1-x}{\sigma_1-x+1} & \text{for } x < \sigma_1(t). \end{cases} \tag{2.3}$$

This function is bounded by a Lebesgue integrable function H

$$|F(t,x,y)| \le H(t) \quad \text{for a.e. } t \in J, \text{ for all } (x,y) \in R^2. \tag{2.4}$$

Define a function $\beta : R \to R$

$$\beta(y) = \begin{cases} y & \text{if } |y| \le K, \\ K \cdot \operatorname{sign} y & \text{if } |y| > K \end{cases}$$

and

$$\begin{aligned} K = \max\Big\{\Big|\phi^{-1}\Big(-\max\{\big|\phi(-\frac{r}{\delta})\big|, \big|\phi(\frac{r}{\delta})\big|\} - \|H\|_{L(J)}\Big)\Big|, \\ \Big|\phi^{-1}\Big(\max\{\big|\phi(-\frac{r}{\delta})\big|, \big|\phi(\frac{r}{\delta})\big|\} + \|H\|_{L(J)}\Big)\Big|\Big\} + \|\sigma_1'\|_\infty + \|\sigma_2'\|_\infty, \end{aligned} \tag{2.5}$$

where

$$r = \|\sigma_1\|_\infty + \|\sigma_2\|_\infty, \quad \delta = \min_{j \in \{0,\dots,m\}} (t_{j+1} - t_j).$$

We consider the following modified problem

$$\frac{d}{dt}[\phi(x'(t))] = F(t,x(t),x'(t)), \tag{2.6}$$

$$\begin{aligned} x(0) &= \varphi(0, x(0) + x'(0) - x'(T)), \\ x(T) &= \varphi(0, x(0) + x'(0) - x'(T)), \end{aligned} \tag{2.7}$$

$$x(t_i+) = x(t_i) - \varphi(t_i, x(t_i)) + J_i(\varphi(t_i, x(t_i))) = \widetilde{J}_i(x(t_i)), \quad i = 1, \dots m, \tag{2.8}$$

$$\phi(x'(t_i+)) - \phi(x'(t_i)) = \phi\big(M_i(\beta(x'(t_i)))\big) - \phi\big(\beta(x'(t_i))\big), \quad i = 1, \dots m. \tag{2.9}$$

For this problem the following three lemmas rule

Lemma 2.1 *Let u be a solution of (2.6)–(2.9) and (0.4), (0.6) hold. Let σ_1, σ_2 be respectively lower and upper functions of (0.1)–(0.3) and $\sigma_1 \le \sigma_2$ on J. Then u satisfies*

$$\sigma_1(t) \le u(t) \le \sigma_2(t) \quad \text{for all } t \in J. \tag{2.10}$$

Proof We show that $v(t) = \sigma_1(t) - u(t) \le 0$ for all $t \in J$. By (2.7), we have $v(0) = v(T) < 0$.

1. Assume, on the contrary, that there is $\alpha \in (0,T)\backslash\Delta$ such that

$$\max\{(\sigma_1 - u)(t) : t \in J\} = v(\alpha) > 0.$$

Then $(\sigma_1 - u)'(\alpha) = 0$. This guarantees the existence of $\delta > 0$ such that

$$(\sigma_1 - u)(t) > 0, \quad |v'(t)| < \frac{\sigma_1 - u}{\sigma_1 - u + 1} < 1 \quad \forall t \in (\alpha, \alpha+\delta) \subset (0,T)\backslash\Delta. \tag{2.11}$$

Using (2.3), (2.11) and the properties of σ_1, we get

$$\begin{aligned} &[\phi(\sigma_1'(t))]' - [\phi(u'(t))]' \\ &\ge f(t,\sigma_1(t),\sigma_1'(t)) - f(t,\sigma_1(t),u'(t)) + \omega_1\Big(t, \frac{\sigma_1(t)-u(t)}{\sigma_1(t)-u(t)+1}\Big) + \frac{\sigma_1(t)-u(t)}{\sigma_1(t)-u(t)+1} \\ &> -|f(t,\sigma_1(t),\sigma_1'(t)) - f(t,\sigma_1(t),u'(t))| + \omega_1(t, |\sigma_1'(t) - u'(t)|) + |\sigma_1'(t) - u'(t)| > 0 \end{aligned}$$

for a.e. $t \in (\alpha, \alpha+\delta)$.

Hence, $\phi(\sigma_1'(t)) - \phi(u'(t)) > \phi(\sigma_1'(\alpha)) - \phi(u'(\alpha)) = 0$ for all $t \in (\alpha, \alpha+\delta)$. Since ϕ is increasing, we get $u'(t) < \sigma_1'(t)$ for all $t \in (\alpha, \alpha+\delta)$. This contradicts that v has a maximum at α. We have showed that v does not have a positive maximum at any point of $(0,T)\backslash\Delta$.

2. If $v(t) > 0$ for some $t \in J$, there is a $t_j \in \Delta$ such that

$$\max\{v(t) : t \in [0,T]\} = v(t_j) > 0. \tag{2.12}$$

By (2.8) and the Definition 2 we get

$$v(t_j+) = \sigma_1(t_j+) - u(t_j+) = J_j(\sigma_1(t_j)) - u(t_j) + \sigma_1(t_j) - J_j(\sigma_1(t_j)) = v(t_j).$$

Then

$$v'(t_j+) \le 0. \tag{2.13}$$

Futhermore, taking into account (2.12), we have $v'(t_j) \ge 0$, and by Definition 2, the relations

$$\begin{gathered} \phi(\sigma_1'(t_j+)) \ge \phi\big(M_j(\sigma_1'(t_j))\big) \ge \phi\big(M_j(\beta(u'(t_j)))\big) \\ = \phi(u'(t_j+)) - \phi(u'(t_j)) + \phi(\beta(u'(t_j))) \ge \phi(u'(t_j+)) \\ \Rightarrow \phi(\sigma_1'(t_j+)) - \phi(u'(t_j+)) \ge 0 \end{gathered}$$

follow. It means, since a function ϕ is increasing,

$$v'(t_j+) \ge 0. \tag{2.14}$$

Now, by (2.13), (2.14) we get $v'(t_j+) = 0$.

Thus, in view of the first part of the proof, there is $\delta > 0$ such that

$$v(t) > 0, \quad |v'(t)| < \frac{\sigma_1 - u}{\sigma_1 - u + 1} < 1 \quad \text{on } (t_j, t_j + \delta) \subset (0,T)\backslash\Delta$$

and we deduce that $v'(t) > 0$ for all $t \in (t_j, t_j + \delta)$, which contradicts (2.12). So, we have proved $\sigma_1(t) \le u(t)$ for all $t \in J$.

If we put $v(t) = u(t) - \sigma_2(t)$, we can prove $u(t) \le \sigma_2(t)$ on J by an analogous argument. □

Lemma 2.2 *Let u be a solution of (2.6)–(2.9) with a condition (0.6). Then u satisfies the periodic boundary conditions (0.2).*

Proof The first, we prove

$$\sigma_1(0) \le u(0) + u'(0) - u'(T) \le \sigma_2(0). \tag{2.15}$$

Suppose, on the contrary, that

$$u(0) + u'(0) - u'(T) > \sigma_2(0). \tag{2.16}$$

By the definition of the function φ it follows that $\varphi(0, u(0) + u'(0) - u'(T)) = \sigma_2(0)$. Then, by condition (2.7), we get $\sigma_2(0) = u(0)$. The inequality (2.16) implies that

$$u'(0) > u'(T). \tag{2.17}$$

The equality $\sigma_2(0) = u(0) = u(T) = \sigma_2(T)$ and (2.10)) yield $\sigma_2'(0) \ge u'(0)$ and $\sigma_2'(T) \le u'(T)$. This together with Definition 2, this head to

$$u'(0) \le \sigma_2'(0) \le \sigma_2'(T) \le u'(T),$$

contrary to (2.17). We can similary derive the inequality $\sigma_1(0) \le u(0) + u'(0) - u'(T)$.

So, if (2.15) is valid, then

$$u(0) = \varphi(0, u(0) + u'(0) - u'(T)) = u(0) + u'(0) - u'(T) \Rightarrow u'(0) = u'(T).$$

It means that a solution of (2.6)–(2.9) fulfils periodic boundary conditions. □

Lemma 2.3 *Let u be a solution of (2.6)–(2.9) with a condition (0.6). Then u satisfies the impulsive conditions (0.3).*

Proof By means of Lemma 2.1 the equality $\varphi(t_i, u(t_i)) = u(t_i)$ holds. Then the condition (2.8) implies $u(t_i+) = J_i(u(t_i))$ for all $i \in \{1, \ldots, m\}$. We will prove the impulsive condition for u'.
We show that

$$\phi(M_j(u'(t_j))) = \phi\big(M_j(\beta(u'(t_j)))\big), \quad \phi(u'(t_j)) = \phi\big(\beta(u'(t_j))\big) \quad \forall t_j \in \Delta.$$

By the Mean Value Theorem there exists $\xi_j \in (t_j, t_{j+1})$, $j = 0, \dots, m$, such that

$$|u'(\xi_j)| = \frac{|u(t_{j+1}) - u(t_j)|}{t_{j+1} - t_j} \le \frac{r}{\delta}.$$

Then the equality

$$u'(t_j) = \phi^{-1}\big(\phi(u'(\xi_j)) + \int_{\xi_j}^{t_j} [\phi(u'(s))]' ds\big).$$

holds for all $j \in \{1, \dots, m\}$. With respect to (2.4), (2.5) and (2.6) we have

$$|u'(t_j)| \le K, \quad j = 1, \dots, m.$$

By (2.9), it means that u fulfils

$$\phi(u'(t_i+)) - \phi(u'(t_i)) = \phi(M_i(u'(t_i))) - \phi(u'(t_i)) \quad \forall i \in \{1, \dots, m\},$$

therefore $u'(t_i+) = M_i(u'(t_i))$ for all $i \in \{1, \dots, m\}$, which concludes the proof. □

Now, we will prove the main result of this section concerning the existence of a solution for problem (0.1)–(0.3) with a bounded right-hand side.

Theorem 2.1 *Let σ_1, σ_2 be respectively lower and upper functions of the problem (0.1)–(0.3) and $\sigma_1 \le \sigma_2$ on J.*

Assume that (0.4) and (0.6) hold. Further assume that there is $h \in L(J)$ such that $|f(t, x, y)| \le h(t)$ for a.e. $t \in J$, for all $(x, y) \in [\sigma_1(t), \sigma_2(t)] \times R$. Then the problem (0.1)–(0.3) has a solution u fulfilling

$$\sigma_1 \le u \le \sigma_2 \quad \text{on } J. \tag{2.10}$$

Proof By means of the three previous lemmas it is sufficient to prove the existence of a solution of the auxiliary problem (2.6)–(2.9). Denote

$$\Psi_u(t) = \sum_{i=1}^{m} \chi_{(t_i,T]}(t)\big[\phi\big(M_i(\beta(u'(t_i)))\big) - \phi\big(\beta(u'(t_i))\big)\big] \quad \text{for } t \in J, \tag{2.18}$$

where $\chi_{(t_j,T]}(t)$ means the characteristic function of the interval $(t_j, T]$. For fixed $v \in C^1_\Delta(J)$ define $g_v : R \to R$ such that

$$g_v(x) = \int_0^T \phi^{-1}\big(x + \int_0^r F_v(s)ds + \Psi_u(r)\big)dr \quad \forall x \in R,$$

where $F_v(s) \equiv F(s, v(s), v'(s))$ for a.e. $s \in J$. Since ϕ^{-1} is continuous and increasing, g_v is continuous and increasing, too. We know that there is $H \in L(J)$ such that $|F_v(s)| \le H(s)$ for a.e. $s \in J$ and for all $v \in C^1_\Delta(J)$ and then

$$|\int_0^t F_v(s)| \le \|H\|_{L(J)} \quad \text{for all } t \in J \text{ and every } v \in C^1_\Delta(J). \tag{2.19}$$

By (2.18), there exists $\varrho > 0$ such that

$$\big|\Psi_u(t)\big| \le \varrho \quad \forall t \in J,\ u \in C^1_\Delta(J). \tag{2.20}$$

Since ϕ is increasing, for each $x \in R$ and for all $v \in C^1_\Delta(J)$

$$T\phi^{-1}(x - \|H\|_{L(J)} - \varrho) \le g_v(x) \le T\phi^{-1}(x + \|H\|_{L(J)} + \varrho).$$

holds. By this inequalities and by the fact that $\phi^{-1}(R) = R$, we have $g_v(R) = R$ for each $v \in C^1_\Delta(J)$. Therefore, for all $v \in C^1_\Delta(J)$ there exists a unique A_v satisfying

$$g_v(A_v) = \int_0^T \phi^{-1}\big(A_v + \int_0^r F_v(s)ds + \Psi_v(r)\big)dr = -\sum_{i=1}^m \big(\widetilde{J}_i(u(t_i)) - u(t_i)\big). \tag{2.21}$$

We show that there exists $N > 0$ such that $|A_v| \le N$ for every $v \in C^1_\Delta(J)$. The Mean Value Theorem for integrals implies that there is $\eta \in (0, T)$ such that

$$\begin{aligned}&\int_0^T \phi^{-1}\big(A_v + \int_0^r F_v(s)ds + \Psi_v(r)\big)dr \\ &= T\phi^{-1}\Big(A_v + \int_0^\eta F_v(s)ds + \Psi_v(\eta)\Big) = -\sum_{i=1}^m \Big(\widetilde{J}_i(u(t_i)) - u(t_i)\Big) = C.\end{aligned}$$

Then $A_v = \phi(\frac{C}{T}) - \int_0^\eta F_v(s)ds - \Psi_v(\eta)$ and

$$\begin{aligned}|A_v| &= \Big|\phi\Big(\frac{C}{T}\Big) - \int_0^\eta F_v(s)ds - \Psi_v(\eta)\Big| \le \Big|\phi\Big(\frac{C}{T}\Big)\Big| + \int_0^\eta |F_v(s)|ds + \big|\Psi_v(\eta)\big| \\ &\le \Big|\phi\Big(\frac{C}{T}\Big)\Big| + \int_0^T H(s)ds + \varrho = \Big|\phi\Big(\frac{C}{T}\Big)\Big| + \|H\|_{L(J)} + \varrho.\end{aligned}$$

It means that

$$|A_v| \le \Big|\phi\Big(\frac{C}{T}\Big)\Big| + \|H\|_{L(J)} + \varrho = N \quad \text{for all } v \in C^1_\Delta(J). \tag{2.22}$$

Now define the following operator $\mathcal{T} : C^1_\Delta(J) \to C^1_\Delta(J)$ by the formula

$$\begin{aligned}(\mathcal{T}u)(t) = &\sum_{i=1}^m \chi_{(t_i,T]}(t)\big(\widetilde{J}_i(u(t_i)) - u(t_i)\big) + \varphi(0, u(0) + u'(0) - u'(T)) \\ &+ \int_0^t \phi^{-1}\big(A_u + \int_0^r F_u(s)ds + \Psi_u(r)\big)dr.\end{aligned} \tag{2.23}$$

Then for all $t \in J$ and all $u \in C^1_\Delta(J)$

$$(\mathcal{T}u)'(t) = \phi^{-1}\big(A_u + \int_0^t F_u(s)ds + \Psi_u(t)\big) \tag{2.24}$$

holds. If $u \in C^1_\Delta(J)$ is a fixed point of $\mathcal{T}$, then from equation (2.24), we obtain

$$\phi(u'(t)) = A_u + \int_0^t F_u(s)ds + \Psi_u(t) \text{ for all } t \in J \text{ and for every } u \in C^1_\Delta(J). \tag{2.25}$$

$F \in Car(J \times R^2)$ means that $F_u \in L(J)$, so we have $\phi(u') \in AC_\Delta(J)$. Differentiating in equation (2.25), we obtain that u satisfies equation (2.6). Using (2.21) we see that u satisfies conditions (2.7). From equation (2.25) we get for all $j \in \{1, \dots, m\}$ equalities

$$\phi(u'(t_j)) = A_u + \int_0^{t_j} F_u(s)ds + \sum_{i=1}^{j-1} \chi_{(t_i,T]}(t)\big[\phi\big(M_i(\beta(u'(t_i)))\big) - \phi\big(\beta(u'(t_i))\big)\big],$$

$$\phi(u'(t_j+)) = A_u + \int_0^{t_j} F_u(s)ds + \sum_{i=1}^{j} \chi_{(t_i,T]}(t)\big[\phi\big(M_i(\beta(u'(t_i)))\big) - \phi\big(\beta(u'(t_i))\big)\big].$$

From the difference of the left-hand and right-hand sides of these equalities we see that for all $t_j \in \Delta$ condition (2.9) follows. Moreover, from equation (2.23) we deduce

$$u(t_j+) = \widetilde{J}_j(u(t_j)) \quad \text{for every } j \in \{1, \dots, m\}.$$

Thus, if u is a fixed point of the operator $\mathcal{T}$ then u is a solution of (2.6)–(2.9).

Now, we will prove that the operator $\mathcal{T}$ has a fixed point $u \in C^1_\Delta(J)$. We start showing that the operator $\mathcal{T}$ is continuous in $C^1_\Delta(J)$. For $\{u_n\} \subset C^1_\Delta(J)$, we prove

$$u_n \to u \text{ in } C^1_\Delta(J) \implies \mathcal{T}u_n \to \mathcal{T}u \text{ in } C^1_\Delta(J).$$

Let A_n corrензpond to u_n by equation (2.21), and similarly let A correnspond to u. We prove that $A_n \to A$. By the construction of A_n and A and by the Mean Value Theorem there exists $\xi_n \in (0,T)$ such that

$$\begin{aligned}&\lim_{n\to\infty}\Big\{\int_0^T \phi^{-1}\big(A_n + \int_0^r F_{u_n}(s)ds + \Psi_{u_n}(r)\big)dr - \int_0^T \phi^{-1}\big(A + \int_0^r F_u(s)ds + \Psi_u(r)\big)dr\Big\}\\ &= T\lim_{n\to\infty}\Big\{\phi^{-1}\big(A_n + \int_0^{\xi_n} F_{u_n}(s)ds + \Psi_{u_n}(\xi_n)\big) - \phi^{-1}\big(A + \int_0^{\xi_n} F_u(s)ds + \Psi_u(\xi_n)\big)\Big\} = 0.\end{aligned} \tag{2.26}$$

Since ϕ is uniformly continuous in J, we have

$$\lim_{n\to\infty}\Big\{A_n + \int_0^{\xi_n} F_{u_n}(s)ds + \Psi_{u_n}(\xi_n) - A - \int_0^{\xi_n} F_u(s)ds - \Psi_u(\xi_n)\Big\} = 0.$$

By the continuity of ϕ and β in u it follows that $\|\Psi_{u_n} - \Psi_u\|_\infty \to 0$. Since $u_n \to u$ in $C^1_\Delta(J)$ and $F \in Car(J \times R^2)$, it holds that $F_{u_n} \to F_u$ a.e on J. By the Lebesgue theorem and from (2.19) we have

$$\lim_{n\to\infty} \int_0^{\xi_n} \big[F_{u_n}(s) - F_u(s)\big]ds = 0.$$

We conclude that $\lim_{n\to\infty} A_n = A$. Furthermore

$$A_n + \int_0^t F_{u_n}(s)ds + \Psi_{u_n}(t) \to A + \int_0^t F_u(s)ds + \Psi_u(t) \quad \text{for all } t \in J.$$

Now, since

$$\Big|A_n + \int_0^t F_{u_n}(s)ds + \Psi_{u_n}(t) - A - \int_0^t F_u(s)ds - \Psi_u(t)\Big| \\ \le |A_n - A| + \|F_{u_n} - F_u\|_{L(J)} + \|\Psi_{u_n} - \Psi_u\|_\infty,$$

for all $t \in J$, the convergence is uniform. By the uniform continuity ϕ^{-1} on compact intervals, $(\mathcal{T}u_n)' \to (\mathcal{T}u)'$ uniformly on J.

Since φ is continuous

$$\varphi(0, u_n(0) + u_n'(0) - u_n'(T)) \to \varphi(0, u(0) + u'(0) - u'(T))$$

in R. Since $\widetilde{J}_i$ are continuous for all $i \in \Delta$

$$\sum_{i=1}^m \chi_{(t_i,T]}(\cdot)\big(\widetilde{J}_i(u_n(t_i)) - u_n(t_i)\big) \to \sum_{i=1}^m \chi_{(t_i,T]}(\cdot)\big(\widetilde{J}_i(u(t_i)) - u(t_i)\big)$$

uniformly on J. Thus $\mathcal{T}u_n \to \mathcal{T}u$ uniformly on J.

Now, we are going to prove a compactness of the operator $\mathcal{T}$. Let M be an arbitrary set in $C^1_\Delta(J)$ and $\{x_n\} \subset \overline{\mathcal{T}(M)}$ be an arbitrary sequence. We prove that we can choose a subsequence convergent in $C^1_\Delta(J)$ to the function $x \in \overline{\mathcal{T}(M)}$. Choose sequence $\{x_n\} \subset \overline{\mathcal{T}(M)}$. Then

$$x_n(t) = \begin{cases} x_n^{[0]}(t), & t \in [0, t_1], \\ x_n^{[1]}(t), & t \in (t_1, t_2], \\ \quad \cdots\cdots\cdots\cdots \\ x_n^{[m]}(t), & t \in (t_m, T], \end{cases}$$

where $\{x_n^{[i]}\} \subset C^1[t_i, t_{i+1}], i = 0, \ldots, m$. Consider $\{x_n^{[0]}\} \subset C^1[0, t_1]$. We will show that this sequence is bounded and $\{(x_n^{[0]})'\}$ is equicontinuous on $[0, t_1]$. Let $u_n \in M$ be such that $x_n = \mathcal{T}u_n$. Then by (2.19), (2.20) and (2.22)

$$\|x_n^{[0]}\|_{C^1[0,t_1]} \le \sum_{i=1}^m |\widetilde{J}_i(u(t_i)) - u(t_i)| + \|\sigma_1\|_\infty + \|\sigma_2\|_\infty \\ + \int_0^t \Big|\phi^{-1}\big(A_{u_n} + \int_0^r F_{u_n}(s)ds + \Psi_{u_n}(r)\big)\Big|dr + \Big|\phi^{-1}\big(A_{u_n} + \int_0^t F_{u_n}(s)ds + \Psi_{u_n}(t)\big)\Big| \\ \le \sum_{i=1}^m |\widetilde{J}_i(u(t_i)) - u(t_i)| + \|\sigma_1\|_\infty + \|\sigma_2\|_\infty \\ + (T+1)\max\{|\phi^{-1}(-N - \|H\|_{L(J)} - \varrho)|, |\phi^{-1}(N + \|H\|_{L(J)}) + \varrho)|\}.$$

It means that $\{x_n^{[0]}\}$ is bounded.

On the basis of the absolute continuity of the Lebesgue integral the condition

$$\forall\varepsilon_1>0\ \exists\delta_1>0\ \forall\tau_1,\tau_2\in[0,t_1]\ \forall x_n\in\overline{\mathcal{T}(M)}:|\tau_1-\tau_2|<\delta_1$$
$$\Rightarrow\Big|A_{u_n}+\int_0^{\tau_1}F_{u_n}(s)ds+\Psi_{u_n}(t_1)-\Big(A_{u_n}+\int_0^{\tau_2}F_{u_n}(s)ds+\Psi_{u_n}(t_1)\Big)\Big|$$
$$=\Big|\int_{\tau_1}^{\tau_2}F_{u_n}(s)\,ds\Big|<\Big|\int_{\tau_1}^{\tau_2}H(s)\,ds\Big|<\varepsilon_1 \tag{2.27}$$

holds. By the uniform continuity of ϕ^{-1} we have

$$\forall\varepsilon>0\ \exists\varepsilon_2>0\ \forall\tau_1,\tau_2\in[0,t_1]\ \forall x_n\in\overline{\mathcal{T}(M)}:$$
$$\Big|A_{u_n}+\int_0^{\tau_1}F_{u_n}(s)\,ds+\Psi_{u_n}(t_1)-\Big(A_{u_n}+\int_0^{\tau_2}F_{u_n}(s)\,ds+\Psi_{u_n}(t_1)\Big)\Big|<\varepsilon_2$$
$$\Longrightarrow\Big|\phi^{-1}\Big(A_{u_n}+\int_0^{\tau_1}F_{u_n}(s)\,ds+\Psi_{u_n}(t_1)\Big)$$
$$-\phi^{-1}\Big(A_{u_n}+\int_0^{\tau_2}F_{u_n}(s)\,ds+\Psi_{u_n}(t_1)\Big)\Big|<\varepsilon.$$

If we choose δ_2 corrensponding to ε_2 by (2.27), then

$$\forall\varepsilon>0\ \exists\delta_2\ \forall\tau_1,\tau_2\in[0,t_1]\ \forall x_n\in\overline{\mathcal{T}(M)}:|\tau_1-\tau_2|<\delta_2$$
$$\Longrightarrow\big|(x_n^{[0]})'(\tau_1)-(x_n^{[0]})'(\tau_2)\big|=\Big|\phi^{-1}\Big(A_{u_n}+\int_0^{\tau_1}F_{u_n}(s)\,ds+\Psi_{u_n}(t_1)\Big)$$
$$-\phi^{-1}\Big(A_{u_n}+\int_0^{\tau_2}F_{u_n}(s)\,ds+\Psi_{u_n}(t_1)\Big)\Big|<\varepsilon.$$

It means that $\{(x_n^{[0]})'\}$ is equicontinuous. We can do similar considerations for the other sequences $\{x_n^{[i]}\}\subset C^1[t_i,t_{i+1}], i=1,\dots,m$. Now, we select $\{x_n^{[0]}\}\subset\{x_{k_n}^{[0]}\}$ convergent in $C^1[0,t_1]$, and corrensponding subsequences $\{x_{k_n}^{[i]}\}\subset\{x_n^{[i]}\}$, $i=1,\dots,m$. Having $\{x_{k_n}^{[1]}\}$ we can select convergent subsequence. Without loss of generality we denote it $\{x_{k_n}^{[1]}\}$ again, and choose corrensponding $\{x_{k_n}^{[i]}\}$, $i=0,2,\dots,m$. Continuing inductively we choose convergent $\{x_{l_n}^{[m]}\}\subset\{x_n^{[m]}\}$ and corrensponding sequences $\{x_{l_n}^{[i]}\}$, $i=0,\dots,m-1$. If we take

$$x_{l_n}(t)=\begin{cases}x_{l_n}^{[0]}(t), & t\in[0,t_1],\\ x_{l_n}^{[1]}(t), & t\in(t_1,t_2],\\ \dots\dots\dots\dots & \\ x_{l_n}^{[m]}(t), & t\in(t_m,T],\end{cases}$$

we obtain the subsequence $\{x_{l_n}(t)\}\subset\{x_n(t)\}\subset\overline{\mathcal{T}(M)}$, such that $\{x_{l_n}(t)\}$ converges in $C^1_\Delta(J)$. It means that the operator $\mathcal{T}$ is compact.

For all $u\in C^1_\Delta(J)$ the following estimate holds

$$\|\mathcal{T}u\|_{C^1_\Delta(J)}\le\sum_{i=1}^m|\widetilde{J}_i(u(t_i))-u(t_i)|+\|\sigma_1\|_\infty+\|\sigma_2\|_\infty$$
$$+(T+1)\max\{|\phi^{-1}(-N-\|H\|_{L(J)}-\varrho)|,|\phi^{-1}(N+\|H\|_{L(J)})+\varrho)|\}=Q.$$

Define $\Omega = \{u \in C^1_\Delta(J) : \|u\|_{C^1_\Delta(J)} \le Q\}$. Then Ω is a nonempty closed bounded and convex set. The operator $\mathcal{T}$ sends the set Ω into Ω, $\mathcal{T}$ is compact. By the Schauder fixed point theorem, operator $\mathcal{T}$ has a fixed point u. This fixed point is a solution of the problem (0.1)–(0.3). □

3 Proofs of main results

In this section we prove the existence results which are contained in Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1 Define function $\psi(t,y) : J \times R \to R$

$$\psi(t,y) = \begin{cases} \varphi_2(t) & \text{if } y > \varphi_2(t), \\ y & \text{if } \varphi_1(t) \le y \le \varphi_2(t), \\ \varphi_1(t) & \text{if } y < \varphi_1(t). \end{cases} \tag{3.1}$$

Further define function $g : J \times R^2 \to R$ by the formula

$$g(t,u,v) = f(t,u,\psi(t,v)) + \frac{v - \psi(t,v)}{|v - \psi(t,v)| + 1}. \tag{3.2}$$

Then there exists $h_0 \in L(J)$

$$|g(t,x,y)| \le h_0(t) \quad \text{for a.e. } t \in J, \text{ for all } (x,y) \in [\sigma_1(t), \sigma_2(t)] \times R.$$

Functions σ_1 and σ_2 are respectively lower and upper functions of the auxiliary problem

$$\frac{d}{dt}[\phi(x'(t))] = g(t,x(t),x'(t)), \tag{3.3}$$

$$x(0) = x(T), \quad \psi(0,x'(0)) = x'(T), \tag{3.4}$$

$$x(t_i+) = J_i(x(t_i)), \quad i \in \{1,\dots,m\}, \tag{3.5}$$

$$x'(t_i+) = x'(t_i) - \psi(t_i,x'(t_i)) + M_i(\psi(t_i,x'(t_i))) = \widetilde{M}_i(x'(t_i)), \ i \in \{1,\dots,m\}, \tag{3.6}$$

function $\widetilde{M}_i$ satisfies condition (0.6) for all $i \in \Delta$. Consider function φ defined by (2.1), further formulas (2.2) - (2.5) defined for function g. By means of the proof of Theorem 2.1 there exists a solution u of the following problem

$$\begin{aligned}
&\frac{d}{dt}[\phi(x'(t))] = F(t,x(t),x'(t)),\\
&x(0) = \varphi(0,x(0) + \psi(0,x'(0)) - x'(T)),\\
&x(T) = \varphi(0,x(0) + \psi(0,x'(0)) - x'(T)),\\
&x(t_i+) = x(t_i) - \varphi(t_i,x(t_i)) + J_i(\varphi(t_i,x(t_i))) = \widetilde{J}_i(x(t_i)), \quad i = 1,\dots m,\\
&\phi(x'(t_i+)) - \phi(x'(t_i)) = \phi\big(\widetilde{M}_i(\beta(x'(t_i)))\big) - \phi\big(\beta(x'(t_i))\big), \quad i = 1,\dots m.
\end{aligned}$$

with a property $\sigma_1 \le u \le \sigma_2$ on J.

In additions, function u is also solution of the problem (3.3)–(3.6). We will show that the following inequalities hold

$$\varphi_1 \le u' \le \varphi_2 \quad \text{on } J. \tag{3.7}$$

Since ϕ is increasing, it is enough to prove the inequality $\phi(\varphi_1) \le \phi(u') \le \phi(\varphi_2)$ on J.

1. Put $z = \phi(u') - \phi(\varphi_2)$ on J. Assume, that there is $\alpha \in (0,T)\backslash\Delta$ such that z has a positive local maximum at α, i.e. $z(\alpha) > 0$. Since u is a solution of the problem (3.3) - (3.6), there is $\delta > 0$ such that $z(t) > 0$ on $(\alpha, \alpha+\delta)$ and

$$\begin{aligned} z'(t) &= [\phi(u'(t))]' - [\phi(\varphi_2(t))]' = g(t, u(t), u'(t)) - [\phi(\varphi_2(t))]' \\ &\ge f(t, u(t), \varphi_2(t)) + \frac{u' - \varphi_2(t)}{u' - \varphi_2(t) + 1} - f(t, u(t), \varphi_2(t)) > 0 \end{aligned}$$

holds for a.e. $t \in (\alpha, \alpha+\delta)$ with respect to (1.8). Thus, for a.e. $t \in (\alpha, \alpha+\delta)$ we have $z'(t) > 0$. By integration of this inequality we get

$$\begin{aligned} 0 < \int_\alpha^t z'(s)ds &= \int_\alpha^t ([\phi(u'(s))]' - [\phi(\varphi_2(s))]')\, ds \\ = \phi(u'(t)) - \phi(\varphi_2(t)) &- (\phi(u'(\alpha)) - \phi(\varphi_2(\alpha))) = z(t) - z(\alpha). \end{aligned}$$

It means that $z(t) > z(\alpha)$ for all $t \in (\alpha, \alpha+\delta)$. It contradicts the assumption of the local maximum of z in α.

2. Assume that there is $t_j \in \Delta$ such that $z(t_j) > 0$. Then $u'(t_j) > \varphi_2(t_j)$. Since

$$(u' - \varphi_2)(t_j+) \ge u'(t_j) - \varphi_2(t_j) + M_j(\varphi_2(t_j)) - M_j(\varphi_2(t_j)) > 0,$$

the inequality $z(t_j+) > 0$ holds. Then there exists $\delta > 0$ such that

$$z(t) > 0 \ \text{ on } (t_j, t_j+\delta), \quad z'(t) > 0 \ \text{ for a.e. } t \in (t_j, t_j+\delta). \tag{3.8}$$

By the first part of the proof we have

$$z'(t) \ge 0 \quad \text{on } (t_j, t_{j+1}). \tag{3.9}$$

Now, by (3.8) and (3.9) we obtain

$$\max_{t \in (t_j, t_{j+1}]} z(t) = z(t_{j+1}) > 0.$$

Continuing inductively we get $z(T) = \phi(u'(T)) - \phi(\varphi_2(T)) > 0$. It means that $u'(T) > \varphi_2(T) \ge \varphi_2(0)$. It is contradiction because from (1.7) and (3.4) we get $u'(T) \le \varphi_2(0) \le \varphi_2(T)$. It means that the inequality $u' \le \varphi_2$ holds on J. By an analogous argument we can prove inequality $\varphi_1 \le u'$ using function $z(t) = \phi(\varphi_1(t)) - \phi(u'(t))$. So, u fulfils (3.7), consequently, u is a solution of (0.1)–(0.3) satisfying (1.9). □

Before proving Theorem 1.2, we prove the following lemma where we derive a priori estimates for derivatives of solutions.

Lemma 3.1 *Let σ_1, σ_2 be respectively lower and upper functions of the problem (0.1)–(0.3) and $\sigma_1 \le \sigma_2$ on J. Assume that (0.5) holds. Further assume that $k \in L(J)$ is nonnegative a.e. on $[0,T]$, $\omega \in C([0,\infty))$ is positive on $[0,\infty)$ and*

$$\int_{-\infty}^{\phi(-1)} \frac{ds}{\omega(|\phi^{-1}(s)|)} = \infty, \qquad \int_{\phi(1)}^{\infty} \frac{ds}{\omega(|\phi^{-1}(s)|)} = \infty. \tag{3.10}$$

Then there exists $\mu_* > 0$ such that for each function $u \in C^1_\Delta(J)$ fulfiling (0.2), the conditions for derivative in (0.3) and inequalities

$$\sigma_1 \le u \le \sigma_2 \quad \textit{on } J, \tag{3.11}$$

$$[\phi(u'(t))]' \le \omega(|u'(t)|)(k(t) + |u'(t)|) \quad \textit{for a.e. } t \in J, \tag{3.12}$$

the following estimate holds $|u'(t)| < \mu_*$ for all $t \in J$.

Proof Put $r = \|\sigma_1\|_\infty + \|\sigma_2\|_\infty$. By the Mean Value Theorem there is $\xi_i \in (t_i, t_{i+1})$ such that

$$|u'(\xi_i)| \le \frac{2r}{\delta} + 1 = r_1, \quad i = 0, 1, \ldots, m, \tag{3.13}$$

where

$$\delta = \min_{i=0,1,\ldots,m} (t_{i+1} - t_i).$$

The assumption (3.10) implies the existence of an increasing sequence $\{\mu_j\}_{j=1}^{2m+4} \in (r_1, \infty)$ such that

$$r_1 < M_j(\mu_j) < \mu_{j+1}, \quad -\mu_{m+4+j} < M^{-1}_{m+1-j}(-\mu_{m+3+j}) < -r_1$$

for $j = 1, \ldots, m$ and satisfying

$$\int_{\phi(r_1)}^{\phi(\mu_1)} \frac{ds}{\omega(|\phi^{-1}(s)|)} > r + \|k\|_{L(J)},$$

$$\int_{\phi(\mu_{m+1})}^{\phi(\mu_{m+2})} \frac{ds}{\omega(|\phi^{-1}(s)|)} > r + \|k\|_{L(J)},$$

$$\int_{\phi(-\mu_{m+3})}^{\phi(-r_1)} \frac{ds}{\omega(|\phi^{-1}(s)|)} > r + \|k\|_{L(J)},$$

$$\int_{\phi(-\mu_{m+4})}^{\phi(-\mu_{m+3})} \frac{ds}{\omega(|\phi^{-1}(s)|)} > r + \|k\|_{L(J)},$$

$$\int_{\phi(M_j(\mu_j))}^{\phi(\mu_{j+1})} \frac{ds}{\omega(|\phi^{-1}(s)|)} > r + \|k\|_{L(J)},$$

$$\int_{\phi(-\mu_{m+4+j})}^{\phi(M^{-1}_{m+1-j}(-\mu_{m+3+j}))} \frac{ds}{\omega(|\phi^{-1}(s)|)} > r + \|k\|_{L(J)}$$

for $j = 1, \dots, m$. We estimate u' from above. Assume that there is $\beta_1 \in (\xi_0, t_1]$ such that

$$\max\{u'(t) : t \in [\xi_0, t_1]\} = u'(\beta_1) = c_1 \ > \ r_1.$$

Then we can find $\alpha_1 \in (\xi_0, \beta_1)$ such that $u'(\alpha_1) = r_1$, $u'(t) > r_1$ for all $t \in (\alpha_1, \beta_1]$. Integrating the inequality

$$\frac{[\phi(u'(t))]'}{\omega(|u'(t)|)} \le (k(t) + |u'(t)|),$$

which holds for a.e. $t \in (\alpha_1, \beta_1)$, we obtain

$$\int_{\alpha_1}^{\beta_1} \frac{[\phi(u'(t))]'dt}{\omega(u'(t))} \le \int_{\alpha_1}^{\beta_1} \big(k(t) + u'(t)\big)dt.$$

Using substitution $s = \phi(u'(t))$ we get that

$$\int_{\alpha_1}^{\beta_1} \frac{[\phi(u'(t))]'dt}{\omega(u'(t))} = \int_{\phi(r_1)}^{\phi(c_1)} \frac{ds}{\omega(\phi^{-1}(s))}.$$

Moreover,

$$\int_{\alpha_1}^{\beta_1} \big(k(t) + u'(t)\big)dt = \int_{\alpha_1}^{\beta_1} k(t)dt + u(\beta_1) - u(\alpha_1) \le \|k\|_{L(J)} + |\sigma_2(\beta_1) - \sigma_1(\alpha_1)|$$
$$\le \|k\|_{L(J)} + (\|\sigma_2\|_{C(J)} + \|\sigma_1\|_{C(J)}) = r + \|k\|_{L(J)}.$$

So we have

$$\int_{\phi(r_1)}^{\phi(c_1)} \frac{ds}{\omega(\phi^{-1}(s))} \le r + \|k\|_{L(J)},$$

which implies that $\phi(c_1) < \phi(\mu_1)$. Since function ϕ is increasing, it means that $c_1 < \mu_1$. Thus $u'(t) < \mu_1$ for all $t \in [\xi_0, t_1]$.

Next assume that there exists $\beta_2 \in (t_1, t_2]$ such that

$$\sup\{u'(t) : t \in (t_1, t_2]\} = u'(\beta_2) = c_2 \ > \ M_1(\mu_1).$$

Then we can find such $\alpha_2 \in (t_1, \beta_2)$ that $u'(\alpha_2) = M_1(\mu_1)$, $u'(t) > M_1(\mu_1)$ for all $t \in (\alpha_2, \beta_2]$. Integrating inequality

$$\frac{[\phi(u'(t))]'}{\omega(|u'(t)|)} \le k(t) + |u'(t)|,$$

which holds for a.e. $t \in (\alpha_2, \beta_2)$, we get

$$\int_{\alpha_2}^{\beta_2} \frac{[\phi(u'(t))]'dt}{\omega(u'(t))} = \int_{\phi(M_1(\mu_1))}^{\phi(c_2)} \frac{ds}{\omega(\phi^{-1}(s))} \le r + \|k\|_{L(J)},$$

so it must be $c_2 < \mu_2$. We have proved that $u'(t) < \mu_2$ for all $t \in [t_1, t_2]$. Continuing inductively over all intervals (t_j, t_{j+1}), we obtain the estimate $u'(t) < \mu_{m+1}$

for all $t \in [t_m, T]$, from this $u'(0) < \mu_{m+1}$ follows. Using the previous procedure we deduce that $u'(t) < \mu_{m+2}$ for all $t \in [0, \xi_0]$.

Similarly we estimate u' from below. Assume that there exists $\beta_{m+3} \in [0, \xi_0)$ such that

$$\min\{u'(t) : t \in [0, \xi_0]\} = u'(\beta_{m+3}) = -c_{m+3} < -r_1.$$

Then we prove that $-c_{m+3} > -\mu_{m+3}$, tj. $u'(t) > -\mu_{m+3}$ on $[0, \xi_0]$, $u'(T) > -\mu_{m+3}$. From the assumption

$$\inf\{u'(t) : t \in (t_m, T]\} = u'(\beta_{m+4}) = -c_{m+4} < -\mu_{m+3}$$

we get $-c_{m+4} > -\mu_{m+4}$, i.e. $-\mu_{m+4} < u'(t)$ for all $t \in [t_m, T]$. Assume that there exists $\beta_{m+5} \in [t_{m-1}, t_m)$ such that

$$\inf\{u'(t) : t \in (t_{m-1}, t_m]\} = u'(\beta_{m+5}) = -c_{m+5} < M_m^{-1}(-\mu_{m+4}).$$

Then we get $-c_{m+5} > -\mu_{m+5}$, i.e. $-\mu_{m+5} < u'(t)$ for all $t \in [t_{m-1}, t_m]$. We can again prove inductively that $-u'(t) > -\mu_{2m+4}$ for every $t \in [\xi_0, t_1]$. If we put $\mu_* = \mu_{2m+4}$, then $\mu_* > \mu_j$ for all $j \in \{1, \ldots, 2m+3\}$ and therefore $|u'(t)| \leq \mu_*$ for all $t \in J$. □

Proof of Theorem 1.2 Define functions

$$\chi(s, r^*) = \begin{cases} 1 & \text{if} \quad 0 \leq s \leq r^*, \\ 2 - \frac{s}{r^*} & \text{if} \quad r^* < s < 2r^*, \\ 0 & \text{if} \quad s \geq 2r^* \end{cases}$$

and

$$g(t, x, y) = \chi(|x| + |y|, r^*) \cdot f(t, x, y),$$

for $t \in J$, $x, y \in R$, where $r^* = \|\sigma_1\|_\infty + \|\sigma_2\|_\infty + \max\{\mu_*,\ \|\sigma_1'\|_\infty,\ \|\sigma_2'\|_\infty\}$ for μ_* given by Lemma 3.1. For $(x, y) \in [\sigma_1(t), \sigma_2(t)] \times R$, the function $g(t, x, y)$ is bounded on J by a Lebesgue integrable function. In addition, σ_1, σ_2 are respectively lower and upper functions of the problem

$$\frac{d}{dt}[\phi(x'(t))] = g(t, x(t), x'(t)), \quad (0.2), (0.3). \tag{3.14}$$

According to Theorem 2.1 there exists a solution u of the problem (3.14) fulfiling $\sigma_1 \leq u \leq \sigma_2$ on J. Moreover,

$$g(t, x, y) =$$
$$= \chi(|x| + |y|, r^*) \cdot f(t, x, y) \leq \chi(|x| + |y|, r^*) \cdot \omega(|y|)(k + |y|) \leq \omega(|y|)(k + |y|)$$

for a.e. $t \in J$, for all $x \in [\sigma_1, \sigma_2]$, every $y \in R$. It means that function g satisfies condition (1.12) which implies that

$$[\phi(u'(t))]' = g(t, u(t), u'(t)) \leq \omega(|u'(t)|)(k(t) + |u'(t)|) \quad \text{for a.e } t \in J.$$

Then, according to Lemma 3.1, $|u'(t)| \leq \mu_*$ holds for all $t \in J$. So $\|u\|_\infty + \|u'\|_\infty <\ r^*$ and $g(t, u, u') = f(t, u, u')$ for a.e. $t \in J$. It means that a solution u of the problem (3.14) is a solution of the problem (0.1)–(0.3), too. It concludes the proof of Theorem 1.2. □

References

[1] Cabada, A., Pouso, L. R.: *Existence results for the problem* $(\phi(u'))' = f(t, u, u')$ *with nonlinear boundary conditions.* Nonlinear Analysis **35** (1999), 221–231.

[2] Cabada, A., Pouso, L. R.: *Existence result for the problem* $(\phi(u'))' = f(t, u, u')$ *with periodic and Neumann boundary conditions.* Nonlinear Anal. T.M.A **30** (1997), 1733–1742.

[3] O'Regan, D.: *Some general principles and results for* $(\phi(u'))' = qf(t, u, u')$*,* $0 < t < 1$. SIAM J. Math. Anal. **24** (1993), 648–668.

[4] Manásevich R., Mawhin, J.: *Periodic solutions for nonlinear systems with p-Laplacian like operators.* J. Differential Equations **145** (1998), 367–393.

[5] Bainov, D., Simeonov, P.: Impulsive differential equations: periodic solutions and applications. *Pitman Monographs and Surveys in Pure and Applied Mathematics 66, Longman Scientific and Technical, Essex, England*, 1993.

[6] Cabada, A., Nieto, J. J., Franco, D., Trofimchuk, S. I.: *A generalization of the monotone method for second order periodic boundary value problem with impulses at fixed points.* Dyn. Contin. Discrete Impulsive Syst. **7** (2000), 145–158.

[7] Yujun Dong, *Periodic solutions for second order impulsive differential systems.* Nonlinear Anal. T.M.A **27** (1996), 811–820.

[8] Erbe, L. H., Xinzhi Liu: *Existence results for boundary value problems of second order impulsive differential equations.* J. Math. Anal. Appl. **149** (1990), 56–59.

[9] Shouchuan Hu, Laksmikantham, V.: *Periodic boundary value problems for second order impulsive differential equations.* Nonlinear Anal. T.M.A **13** (1989), 75–85.

[10] Liz, E., Nieto, J. J.: *Periodic solutions of discontinuous impulsive differential systems.* J. Math. Anal. Appl. **161** (1991), 388–394.

[11] Liz, E., Nieto, J. J.: *The monotone iterative technique for periodic boundary value problems of second order impulsive differential equations.* Comment. Math. Univ. Carolinae **34** (1993), 405–411.

[12] Rachůnková, I., Tomeček, J.: *Impulsive BVPs with nonlinear boundary conditions for the second order differential equations without growth restrictions.* J. Math. Anal. Appl. **292** (2004), 525–539.

[13] Rachůnková, I., Tvrdý, M.: *Impulsive Periodic Boudary Value Problem and Topological Degree.* Funct. Differ. Equ. **9** (2002), 471–498.

[14] Zhitao Zhang: *Existence of solutions for second order impulsive differential equations.* Appl. Math., Ser. B (eng. Ed.) **12** (1997), 307–320.

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On Tensor Fields Semiconjugated with Torse-forming Vector Fields *

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Abstract

The paper deals with tensor fields which are semiconjugated with torse-forming vector fields. The existence results for semitorse-forming vector fields and for convergent vector fields are proved.

Key words: Torse-forming vector fields, Riemannian space, semisymmetric space, T-semisymmetric space.

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1 Introduction

Torse-forming vector fields were introduced by K. Yano [8] in 1944 and their properties in Riemannian spaces have been studied by various mathematicians. For example some properties in Ricci semisymmetric Riemannian spaces have been proved by J. Kowolik in [1]. In T-semisymmetric Riemannian spaces they are studied by the authors in [4] and [5].

This paper is devoted to the study of tensor fields which are semiconjugated with torse-forming vector fields. We are motivated by the work of J. Kowolik [1].

First we give some definitions and notations. V_n denotes an n-dimensional Riemannian space with a metric g and an affine connection ∇. The metric g

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need not be positive definite. TV_n is a space of all tangent vector fields on V_n. In the whole paper we will assume that $n > 2$ and that all functions, vectors and tensor fields are sufficiently smooth. Further $\boldsymbol{\xi}$ will be a non-zero vector field, i.e. $\boldsymbol{\xi}(x) \neq \boldsymbol{o}$ for each $x \in V_n$.

We denote the Riemannian tensor in V_n by R. This tensor is called *harmonic*, if $R^{\alpha}_{ijk,\alpha} = 0$, where "," denotes the covariant derivative. This condition can be written in the form $R_{ij,k} = R_{ik,j}$ where $R_{ij} \equiv R^{\alpha}_{ij\alpha}$ is the Ricci tensor of V_n.

Definition 1 Vector field $\boldsymbol{\xi}$ is called *torse-forming*, if $\nabla_X \boldsymbol{\xi} = \varrho \cdot X + a(X) \cdot \boldsymbol{\xi}$ for all $X \in TV_n$, where ϱ is some function on V_n, a is a linear form on V_n. In the local transcription this formula has the form $\xi^h_{,i} = \varrho\delta^h_i + a_i\xi^h$, where ξ^h are components of the torse-forming field $\boldsymbol{\xi}$, δ^h_i is the Kronecker delta, a_i are components of the form a, which is a covector on V_n.

Definition 2 A torse-forming vector field $\boldsymbol{\xi}$ is called:

- *recurrent*, if $\varrho = 0$,
- *concircular*, if the form a is gradient (or locally gradient), i.e. there exists (locally) a function $\varphi(x)$ such that $a = \partial_i \varphi(x) dx^i$,
- *convergent*, if $\boldsymbol{\xi}$ is concircular and $\varrho = \mathrm{const} \cdot \exp\big(\varphi(x)\big)$,
- *semitorse-forming*, if $R(X, \boldsymbol{\xi})\boldsymbol{\xi} = 0$ for each $X \in TV_n$.

Properties of torse-forming vector fields in the Einsteinian spaces are proved by the authors in [5]. In [2] and [3] J. Mikeš proved that in non-Einsteinian Ricci-symmetric and Ricci-two-symmetric ($R_{ij,kl} = 0$) spaces there are no concircular vector fields which are not recurrent.

In what follows we will need a definition of an operator $R(X,Y) \circ T$ for tensors of the type $(0,q)$ or $(1,q)$.

Let T be a tensor of the type $(0,q)$, which is defined as a q-linear form $T(X_1, X_2, \ldots, X_q)$, where $X_1, X_2, \ldots, X_q \in TV_n$.

In the space V_n we introduce an operator $R(X,Y) \circ T$ in the following way:

$$R(X,Y) \circ T(X_1, X_2, \ldots, X_q) \stackrel{\text{def}}{=} \sum_{s=1}^{q} T(X_1, \ldots, X_{s-1}, R(X,Y)X_s, X_{s+1}, \ldots, X_q).$$

In the local transcription the tensor $R(X,Y) \circ T$ has a form

$$\sum_{s=1}^{q} T_{i_1 \ldots i_{s-1} \alpha i_{s+1} \ldots i_q} R^{\alpha}_{i_s jk}.$$

By the Ricci identity we have

$$T_{i_1 \ldots i_q, [jk]} = \sum_{s=1}^{q} T_{i_1 \ldots i_{s-1} \alpha i_{s+1} \ldots i_q} R^{\alpha}_{i_s jk},$$

where $[jk]$ denotes the alternation of the tensor with respect to j and k.

If T is a tensor of the type $(0,0)$ (i.e. an invariant, which is a function or a scalar on V_n), then we put $R(X,Y)\circ T=0$, or locally $T_{,[jk]}=0$.

Similarly we can define an operator $R(X,Y)\circ T$ for a tensor T of the type $(1,q)$:

$$R(X,Y)\circ T(X_1,X_2,\dots,X_q)$$
$$\stackrel{\text{def}}{=}\sum_{s=1}^{q}T(X_1,\dots,X_{s-1},R(X,Y)X_s,X_{s+1},\dots,X_q)-R(X,Y)\big(T(X_1,\dots,X_q)\big).$$

The tensor $R(X,Y)\circ T$ has a local expression

$$\sum_{s=1}^{q}T^h_{i_1\dots i_{s-1}\alpha i_{s+1}\dots i_q}R^\alpha_{i_sjk}-T^\alpha_{i_1\dots i_q}\cdot R^h_{\alpha jk}.$$

By the Ricci identity we have

$$T^h_{i_1\dots i_q,[jk]}=\sum_{s=1}^{q}T^h_{i_1\dots i_{s-1}\alpha i_{s+1}\dots i_q}R^\alpha_{i_sjk}-T^\alpha_{i_1\dots i_q}\cdot R^h_{\alpha jk}.$$

Now we present Kowolik's theorems of [1] in a modified form which is more convenient for us. These theorems will be generalized in the next parts of our paper. First, recall notions used in the theorems.

Definition 3 A Riemannian space V_n is called *semisymmetric*, if

$$R(X,Y)\circ R=0\quad\forall X,Y\in TV_n. \tag{1}$$

We write (1) locally in the form $R^h_{ijk,[lm]}=0$ or

$$R^h_{\alpha jk}R^\alpha_{ilm}+R^h_{i\alpha k}R^\alpha_{jlm}+R^h_{ij\alpha}R^\alpha_{klm}-R^\alpha_{ijk}R^h_{\alpha lm}=0.$$

Definition 4 A Riemannian space V_n is called *Ricci semisymmetric*, if

$$R(X,Y)\circ Ric=0\quad\forall X,Y\in TV_n. \tag{2}$$

We write (2) locally

$$R_{\alpha j}R^\alpha_{ikl}+R_{i\alpha}R^\alpha_{jkl}=0\quad\text{or}\quad R_{ij,[kl]}=0.$$

Simply conformaly recurrent spaces (*s.c.r. spaces*) were defined by W. Roter [7]. These spaces are characterized by the following conditions:

The Riemannian space V_n is a *s.c.r.* space, if and only if:

1. $C_{hijk}\neq 0$, where C_{hijk} is a Weyl tensor of conformal curvature,
2. $C_{hijk,l}=\varphi_l C_{hijk}$,
3. a vector φ_k is locally gradient,
4. the Ricci tensor is a Codazzi tensor.

Remark 1 It holds that each *s.c.r.* space is semisymmetric.

Theorem 1 ([1]) *Let V_n $(n \geq 4)$ be a Ricci semisymmetric space with a harmonic Riemannian tensor. If there is a torse-forming vector field $\boldsymbol{\xi}$ in V_n, then $\boldsymbol{\xi}$ is either concircular or recurrent.*

Theorem 2 ([1]) *If there is a torse-forming vector field $\boldsymbol{\xi}$ in a s.c.r. space V_n $(n \neq 4)$, then $\boldsymbol{\xi}$ is recurrent.*

Let T be a tensor field of the type $(0,q)$ or $(1,q)$ and $\boldsymbol{\xi}$ be a vector field on V_n. By means of the operator $R(X,\boldsymbol{\xi}) \circ T$ let us define the basic notion of our paper:

Definition 5 The tensor field T is *semiconjugated* with the vector field $\boldsymbol{\xi}$, if

$$R(X,\boldsymbol{\xi}) \circ T = 0 \quad \text{for each } X \in TV_n. \tag{3}$$

In the local transcription (3) has the form

$$T^{\cdot}_{\dots,[lm]}\xi^m = 0, \tag{4}$$

where ξ^m are local components of $\boldsymbol{\xi}$.

2 Vector fields semiconjugated with torse-forming vector fields

In this section we will consider 1-covariant vector fields semiconjugated with a torse-forming vector field $\boldsymbol{\xi}$. Denote by $\xi(X)$ a linear form generated by $\boldsymbol{\xi}$, i.e. $\xi(X) \equiv g(X,\boldsymbol{\xi})$.

Theorem 3 *Let T $(\neq 0)$ be a 1-covariant vector field semiconjugated with a non-isotropic torse-forming vector field $\boldsymbol{\xi}$, which is not convergent. Then $\boldsymbol{\xi}$ is semitorse-forming and T is colinear with a form $\xi(X)$.*

Proof Assume that there is a non-zero vector field T and a non-isotropic non-convergent torse-forming vector field $\boldsymbol{\xi}$, which satisfy (4), i.e.

$$T_\alpha R^\alpha_{ij\beta}\xi^\beta = 0, \tag{5}$$

where T_i are local components of T and R^h_{ijk} are components of the Riemannian tensor R. According to [5] we can assume that $\boldsymbol{\xi}$ is normalized, i.e. $g(\boldsymbol{\xi},\boldsymbol{\xi}) = e = \pm 1$, and the condition

$$\xi_\alpha R^\alpha_{ijk} = g_{ij}c_k - g_{ik}c_j + \xi_i a_{jk} \tag{6}$$

holds, where $a_{jk} \equiv -e\xi_{[j}\varrho_{,k]}$ and

$$c_k \equiv \varrho_{,k} + e\varrho^2\xi_k. \tag{7}$$

Since $\boldsymbol{\xi}$ is not convergent, we have $c_i \neq 0$.

Contracting (6) with $T^k \overset{\text{def}}{=} T_\alpha g^{\alpha k}$ and using (5) and properties of the Riemannian tensor we get

$$g_{ij} c_k T^k - T_i c_j + \xi_i a_{jk} T^k = 0. \tag{8}$$

If $c_k T^k \neq 0$, then (8) gives rank $\|g_{ij}\| \leq 2$. Since $n > 2$, we have $c_k T^k = 0$ and (8) leads to

$$-T_i c_j + \xi_i a_{jk} T^k = 0. \tag{9}$$

Since $c_j \neq 0$, the condition (9) implies

$$T_i = a\xi_i,$$

where a if a non-zero function.

Substituting $T_i = a\xi_i$ in (6) we see, that either $\boldsymbol{\xi}$ is semitorse-forming vector field or $T_i = 0$. This completes the proof of Theorem 3. □

3 Symmetric 2-covariant tensors semiconjugated with a torse-forming vector field

We will prove the following theorem:

Theorem 4 *Let $n > 2$ and let T ($\neq \gamma g$) be a 2-covariant symmetric tensor field semiconjugated with a non-isotropic torse-forming vector field $\boldsymbol{\xi}$, which is not convergent. Then it holds that $\boldsymbol{\xi}$ is semitorse-forming in V_n and*

$$T(X,Y) = \gamma \cdot g(X,Y) + \psi \cdot \xi(X) \cdot \xi(Y) \quad \forall X, Y \in TV_n, \tag{10}$$

where γ, ψ are functions on V_n.

Proof Assume that there is a 2-covariant symmetric tensor field T on V_n, which is semiconjugated with a normalised torse-forming vector field $\boldsymbol{\xi}$, which is not convergent. It means that $\boldsymbol{\xi}$ satisfies (6) and $c_i \neq 0$.

Further we have:

$$R(X, \boldsymbol{\xi}) \circ T = 0 \quad \forall X \in TV_n,$$

i.e. locally

$$T_{\alpha j} R^\alpha_{il\beta} \xi^\beta + T_{i\alpha} R^\alpha_{jl\beta} \xi^\beta = 0. \tag{11}$$

If we substitute (6) in (11) and use properties of the Riemannian tensor we get after computation

$$g_{li} T_{\alpha j} c^\alpha - T_{lj} c_i + g_{lj} T_{i\alpha} c^\alpha - T_{il} c_j + \xi_l \omega_{ij} = 0, \tag{12}$$

where ω is some tensor of the type $(0,2)$ and $c^i \equiv c_\alpha g^{\alpha i}$.

We will prove that

$$T_{\alpha i} c^\alpha = \gamma c_i. \tag{13}$$

Assume, that (13) does not hold. Then there exists a vector ε^i such that

$$c_\alpha \varepsilon^\alpha = 0 \quad \text{and} \quad T_{\alpha\beta}\varepsilon^\alpha c^\beta = 1. \tag{14}$$

Contract (12) with $\varepsilon^i\varepsilon^j$. Since $T_{ij} = T_{ji}$ and (14) holds, we get

$$\varepsilon_l = h\xi_l, \tag{15}$$

where $h \stackrel{\text{def}}{=} -\frac{1}{2}\omega_{\alpha\beta}\varepsilon^\alpha\varepsilon^\beta$.

If we contract (12) with ε^j, we obtain by means of (14) and (15)

$$g_{li} - T_{l\alpha}\varepsilon^\alpha c_i + \xi_l(hT_{i\alpha}c^\alpha + \omega_{i\beta}\varepsilon^\beta) = 0.$$

This implies that rank $\|g_{ij}\| \le 2$, which contradicts the assumption that (13) does not hold.

By (13) we extract the member $T_{\alpha i}c^\alpha$ in (12). After computation we obtain

$$F_{lj}c_i + F_{il}c_j + \xi_l\omega_{ij} = 0, \tag{16}$$

where

$$F_{ij} \stackrel{\text{def}}{=} T_{ij} - \gamma g_{ij}. \tag{17}$$

Since $c_i \ne 0$, then there exists φ^i such, that $c_\alpha\varphi^\alpha = 1$.

Contracting (16) with $\varphi^i\varphi^j$ we get $F_{l\alpha}\varphi^\alpha = f \cdot \xi_l$, where $f \stackrel{\text{def}}{=} -\frac{1}{2}\omega_{\alpha\beta}\varepsilon^\alpha\varepsilon^\beta$.

Similarly, if we contract (16) with φ^j, we get

$$F_{il} = \xi_l\chi_i\,, \tag{18}$$

where $\chi_i \stackrel{\text{def}}{=} -fc_i - \omega_{i\alpha}\varphi^\alpha$.

Since F_{ij} is a symmetric tensor, the equality (18) implies

$$F_{ij} = \psi \cdot \xi_i\xi_j. \tag{19}$$

By the assumption $F_{ij} \ne 0$, we have $\psi \ne 0$. Substituting (17) to (19) we see, that (10) is true. It remains to prove that the vector field $\boldsymbol{\xi}$ is semitorse-forming.

Therefore we covariantly derive the equality (19) by indices l and m, then we alternate it with respect to l and m and finally we contract it with ξ^m. Since

$$F_{ij,[lm]}\xi^m = 0 \quad \text{and} \quad \psi \ne 0,$$

we reach the formula

$$\xi_{i,[lm]}\xi^m \cdot \xi_j + \xi_i \cdot \xi_{j,[lm]}\xi^m = 0,$$

wherefrom it follows

$$\xi_{i,[lm]}\xi^m = 0.$$

This means that the vector field $\boldsymbol{\xi}$ is semitorse-forming. □

4 Antisymmetric 2-covariant tensors semiconjugated with a torse-forming vector field

The following theorem deals with antisymmetric tensor fields.

Theorem 5 *In a Riemannian space V_n $(n > 3)$ there is no non-zero 2-covariant antisymmetric tensor field T semiconjugated with a non-isotropic torse-forming vector field $\boldsymbol{\xi}$, which is not convergent.*

Proof Assume that there is a 2-covariant anti-symmetric tensor field T on V_n, which is semiconjugated with a non-isotropic torse-forming vector field $\boldsymbol{\xi}$, which is not convergent. It means, that $\boldsymbol{\xi}$ satisfies (6) and $c_i \neq 0$. Similarly as in the proof of Theorem 4 we get, that (11), (12) and (13) are true. Substituting (13) in (12) and using the antisymmetric property of T (i.e. $T_{ij} = -T_{ji}$), we get after computation

$$(T_{li} - \mu g_{li})c_j - (T_{lj} - \mu g_{lj})c_i - \xi_l \omega_{ij} = 0. \tag{20}$$

Since $c_j \neq 0$, then there exists φ^i, for which $\varphi^\alpha c_\alpha = 1$. Contracting (20) with φ^j we find

$$T_{li} - \mu g_{li} = \xi_l \eta_i + \chi_l c_i\,, \tag{21}$$

where η_i and χ_l are some covectors.

Symmetrising (21) we obtain

$$-2\mu g_{li} = \xi_l \eta_i + \xi_i \eta_l + \chi_l c_i + \chi_i c_l. \tag{22}$$

If $n > 4$, we deduce that $\mu = 0$.

Assume that $n = 4$ and $\mu \neq 0$. Then covectors ξ_i, c_i, η_i, χ_i must be linearly independent. Hence their coordinates in a given point x can be chosen in the following way:

$$\xi_i = \delta_i^1, \quad \eta_i = \delta_i^2, \quad c_i = \delta_i^3, \quad \chi_i = \delta_i^4.$$

Then

$$g_{ij} = -\frac{1}{2\mu}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The inverse matrix g^{ij} has the form

$$g^{ij} = -2\mu \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

We can check that

$$g^{ij}\xi_i \xi_j = 0$$

holds, i.e. $\boldsymbol{\xi}$ is isotropic, a contradiction.

Thus for $n > 3$ the formula (22) implies, that $\mu = 0$. Therefore we can simplify (21) and (22) as follows:

$$T_{ij} = \xi_i \eta_j + \chi_i c_j$$

and

$$\xi_l \eta_i + \xi_i \eta_l + \chi_l c_i + \chi_i c_l = 0. \tag{23}$$

Vectors ξ_i and χ_i are not colinear. Otherwise it should be $T_{ij} = 0$. Therefore there is φ^i such that

$$\xi_\alpha \varphi^\alpha = 1 \quad \text{and} \quad \chi_\alpha \varphi^\alpha = 0.$$

Contracting (23) with $\varphi^i \varphi^l$ we find $\eta_\alpha \varphi^\alpha = 0$ and contracting (23) with φ^l we get $\eta_i = -c_\alpha \varphi^\alpha \cdot \chi_i$. Then (23) has a form

$$(c_i - c_\alpha \varphi^\alpha \xi_i)\chi_l + (c_l - c_\alpha \varphi^\alpha \xi_l)\chi_i = 0.$$

Since $\chi_l \neq 0$, we obtain

$$c_i = c_\alpha \varphi^\alpha \xi_i. \tag{24}$$

Using (7) and (24) we derive

$$\varrho_{,k} = (c_\alpha \varphi^\alpha - e\varrho^2)\xi_k.$$

Hence we have $\varrho = \varrho(\xi)$, where ξ is a scalar field satisfying $\xi_k = \partial_k \xi$. It means that $\boldsymbol{\xi}$ is concircular and, by [3], is convergent. □

5 Main results

By means of Theorem 4 (for symmetric tensors) and Theorem 5 (for antisymmetric tensors) we will prove the following assertion for arbitrary 2-covariant tensors.

Theorem 6 *Let $n > 3$ and let T $(\neq \gamma g)$ be a 2-covariant tensor field semiconjugated with a non-isotropic torse-forming vector field $\boldsymbol{\xi}$, which is not convergent. Then it holds that $\boldsymbol{\xi}$ is semitorse-forming in V_n and*

$$T(X,Y) = \gamma \cdot g(X,Y) + \psi \cdot \xi(X) \cdot \xi(Y) \quad \forall X, Y \in TV_n,$$

where γ, ψ are functions on V_n.

Proof Assume that there is a 2-covariant tensor field T on V_n, which is semiconjugated with a normalised torse-forming vector field $\boldsymbol{\xi}$, which is not convergent.

Tensor T can be uniquely expressed in the form $T = U + V$, where U is a symmetric part and V is an antisymmetric part of T. It holds

$$U(X,Y) = \frac{1}{2}\big(T(X,Y) + T(Y,X)\big)$$

and

$$V(X,Y) = \frac{1}{2}\big(T(X,Y) - T(Y,X)\big)$$

for any vector fields $X, Y \in TV_n$. Therefore U and V are also semiconjugated with $\boldsymbol{\xi}$. Theorem 5 implies, that $V = 0$. Hence $T \equiv U$ and so T is symmetric and the assertion of Theorem 6 follows from Theorem 4. □

Now we will prove theorems for Riemannian spaces having Riemannian and Ricci tensors semiconjugated with a torse-forming vector field. These theorems generalize Kowolik's results in [1].

Theorem 7 *Let $n > 2$ and let V_n be a non-Einsteinian Riemannian space, where the Ricci tensor is semiconjugated with a non-isotropic torse-forming vector field $\boldsymbol{\xi}$. Then $\boldsymbol{\xi}$ is convergent.*

Proof Assume that the Ricci tensor *Ric* is semiconjugated with a torse-forming vector field $\boldsymbol{\xi}$.

Since *Ric* is a symmetric tensor, we get by Theorem 4

$$Ric(X,Y) = \gamma g(X,Y) + \psi \cdot \xi(X) \cdot \xi(Y) \quad \forall X, Y \in TV_n, \tag{25}$$

where $\xi(X) \stackrel{\text{def}}{=} g(X, \boldsymbol{\xi})$ and ψ is a function on V_n.

Semitorse-forming fields fulfil $R^h_{\alpha j \beta}\xi^\alpha \xi^\beta = 0$. Contracting it with respect to h and j we obtain $R_{\alpha\beta}\xi^\alpha\xi^\beta = 0$, which can be written in the form

$$Ric(\boldsymbol{\xi}, \boldsymbol{\xi}) = 0.$$

Let us put $X = \boldsymbol{\xi}$ a $Y = \boldsymbol{\xi}$ in (25). Since we can assume that $\boldsymbol{\xi}$ is normalized, i.e. $g(\boldsymbol{\xi}, \boldsymbol{\xi}) \equiv \xi(\boldsymbol{\xi}) = e = \pm 1$, we get $\psi = -e\gamma$ and so the formula (25) has the form

$$Ric(X,Y) = \gamma \cdot \big(g(X,Y) - e\xi(X) \cdot \xi(Y)\big) \quad \forall X, Y \in TV_n. \tag{26}$$

Substituting $Y = \boldsymbol{\xi}$ in (26) we obtain

$$Ric(X, \boldsymbol{\xi}) = 0 \quad \forall X \in TV_n.$$

It means that $\boldsymbol{\xi}$ is an eigenvector of the Ricci tensor corresponding to the zero eigenvalue. Therefore $\boldsymbol{\xi}$ is convergent. □

Theorem 8 *Let $n > 2$ and let V_n be a Riemannian space with a non-constant curvature, where the Riemannian tensor is semiconjugated with a non-isotropic torse-forming vector field $\boldsymbol{\xi}$. Then $\boldsymbol{\xi}$ is convergent.*

Proof Assume that a Riemannian space V_n with a non-constant curvature has the Riemannian tensor which is semiconjugated with a torse-forming vector field $\boldsymbol{\xi}$ which is not convergent. Then V_n has the Ricci tensor which is also semiconjugated with $\boldsymbol{\xi}$. Therefore by Theorem 7 the space V_n has to be an Einsteinian space. We can easily see that $\boldsymbol{\xi}$ is concircular.

Then, according to the result of [4] the Riemannian tensor has the form

$$R_{hijk} = K(g_{hj}g_{ik} - g_{hk}g_{ij}),$$

which means that V_n has a constant curvature, a contradiction. We have proved that $\boldsymbol{\xi}$ has to be convergent. □

References

[1] Kowolik, J.: *On some Riemannian manifolds admitting torse-forming vector fields.* Dem. Math. **18**, 3 (1985), 885–891.

[2] Mikeš, J.: *Geodesic mappings of affine-connected and Riemannian spaces* J. Math. Sci. New York **78**, 3 (1996), 311–333.

[3] Mikeš, J.: *Geodesic Ricci mappings of two-symmetric Riemann spaces* Math. Notes **28** (1981), 622–624.

[4] Mikeš, J., Rachůnek L.: *T-semisymmetric spaces and concircular vector fields.* Supplemento ai Rendiconti del Circolo Matematico di Palermo, II. Ser. 69 (2002), 187–193.

[5] Rachůnek, L., Mikeš, J.: *Torse-forming vector fields in T-semisymmetric Riemannian spaces.* Steps in differential geometry. Proceedings of the colloquium on differential geometry, Debrecen, Hungary, July 25–30, 2000. Univ. Debrecen, Institute of Mathematics and Informatics, 2001, 219–229.

[6] Rachůnek, L., Mikeš, J.: *On semitorse-forming vector fields.* 3rd International Conference APLIMAT 2004, Bratislava, 835–840.

[7] Roter W.: *On a class of conformally recurrent manifolds.* Tensor N. S. **39** (1982), 207–217.

[8] Yano, K.: *On torse-forming directions in Riemannian spaces.* Proc. Imp. Acad. Tokyo **20** (1944), 701–705.

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Some Stability and Boundedness Results for the Solutions of Certain Fourth Order Differential Equations

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Abstract

Sufficient conditions are established for the asymptotic stability of the zero solution of the equation (1.1) with $p \equiv 0$ and the boundedness of all solutions of the equation (1.1) with $p \neq 0$. Our result includes and improves several results in the literature ([4], [5], [8]).

Key words: Differential equations of fourth order, boundedness, stability, Lyapunov functions.

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1 Introduction

In the current paper, we consider the nonlinear differential equation of the form

$$x^{(4)} + a(\ddot{x}, \dddot{x})\,\dddot{x} + b(x, \dot{x})\,\ddot{x} + c(\dot{x}) + d(x) = p(t, x, \dot{x}, \ddot{x}, \dddot{x}). \tag{1.1}$$

It can be written in the phase variables form

$$\begin{aligned} \dot{x} &= y, \quad \dot{y} = z, \quad \dot{z} = u, \\ \dot{u} &= -a(z, u)u - b(x, y)z - c(y) - d(x) + p(t, x, y, z, u), \end{aligned} \tag{1.2}$$

in which the functions a, b, c, d and p depend only on the arguments displayed and the dots denote differentiation with respect to t. The functions a, b, c, d and p are continuous for all values of their respective arguments.The derivatives $\frac{\partial a(z,u)}{\partial u} \equiv a_u(z,u)$, $\frac{\partial b(x,y)}{\partial x} \equiv b_x(x,y)$, $\frac{dc}{dy} \equiv c'(y)$, and $\frac{dd}{dx} \equiv d'(x)$ exist and are continuous. Moreover, the existence and the uniqueness of the solutions of (1.1) will be assumed.

It is well known that the stability and boundedness of solutions of ordinary differential equations are very important problems in the theory and applications of differential equations. So far, perhaps, the most effective method to study the stability and boundedness of solutions of nonlinear differential equations is still the Lyapunov's direct (or second) method. In the relevant literature, for the fourth order nonlinear differential equations, many stability and boundedness results have been established by using this method. We refer to [1-8] and the references cited there for some of those topics. In [5], Ponzo discussed the stability of solutions of the equation (1.1) in the case $p(t, x, \dot{x}, \ddot{x}, \dddot{x}) = 0$. Nearly four decades later, Hu [4] proved that the result of Ponzo [5] was not true in general, except the special case $b(x, y) \equiv$ constant and $d(x) \equiv cx$ (c is a constant) in (1.1). Recently, in [8], Wu and Xiong also investigated the asymptotic stability of the zero solution of the differential equations described as follows:

$$x^{(4)} + a_1\dddot{x} + a_2\ddot{x} + a_3\dot{x} + f(x) = 0$$

and

$$x^{(4)} + a_1\dddot{x} + f(x, \dot{x})\ \ddot{x} + a_3\dot{x} + a_4x = 0,$$

in which a_1, a_2, a_3 and a_4 are constants. The motivation for the present work has come from the papers of Ponzo [5], Hu [4], Wu and Xiong [8] and the papers mentioned above. Our aim is to obtain similar results and improve some results in the papers stated above. It should also be noted that the domain of attraction of the zero solution $x = 0$ of the equation (1.1) (for $p \equiv 0$) in the following first result is not going to be determined here.

2 The stability and the boundedness results of solutions of (1.2)

In what follows we shall use the following notations:

$$a_1(z,0) := \begin{cases} \dfrac{1}{z}\displaystyle\int_0^z a(z,0)dz, & z \neq 0 \\ a(0,0), & z = 0 \end{cases}$$

and

$$c_1(y) := \begin{cases} \frac{c(y)}{y}, & y \neq 0 \\ c'(0), & y = 0. \end{cases}$$

For the case $P \equiv 0$ in (1.1) the following result is established.

Theorem 1 *Further to the basic assumptions on the functions* a, b, c *and* d *assume that the following conditions are satisfied* ($\alpha, \beta, \mu, \gamma, \delta, \eta, \varepsilon$ *and* ε_1*—some positive constants):*

(i) $0 \leq a(z,u) - \alpha \leq \varepsilon_1$ *for all* z *and* u.

(ii) $c_1(y) \geq \beta$ *for all* $y \neq 0, c(0) = 0$.

(iii) $0 \leq b(x,y) - \mu \leq \sqrt{\frac{\delta\varepsilon_1}{4\beta}}$ *and*

$$y \int_0^y b_x(x,y) y \, dy \leq -\left(\frac{\beta^2}{\alpha\gamma}\right) y^2$$

for all x *and* y.

(iv) $d(x)x > 0$ *for all* $x \neq 0$, $0 \leq \gamma - d'(x) \leq \frac{\sqrt{\delta}}{2}$ *for all* x, *and* $d(0) = 0$.

(v) $\alpha\beta\mu - \beta c'(y) - \alpha\gamma a(z,u) \geq \delta$ *for all* y, z *and* u.

(vi) $c'(y) - c_1(y) \leq \eta < \frac{2\delta\gamma}{\alpha\beta^2}$ *for all* $y \neq 0$, *and* $a_1(z,u) - a(z,u) \leq \varepsilon < \frac{2\delta}{\alpha^2\beta}$ *for all* $z \neq 0$ *and* u.

(vii) $\gamma y a_u(z,u) + \beta z a_u(z,u) \geq 0$ *for all* y, z *and* u.

Then the trivial solution of the system (1.2) is asymptotically stable.

Remark 1 From the conditions (ii) and (v) of Theorem 1 we can obtain

$$a(z,u) < \frac{\beta\mu}{\gamma} \quad \text{and} \quad c'(y) < \alpha\mu.$$

Remark 2 When $a(\ddot{x}, \dddot{x}) = \alpha$, $b(x,\dot{x}) = \mu$, $c(\dot{x}) = \beta\dot{x}$ and $d(x) = \gamma x$, equation (1.1) reduces to the linear constant coefficient differential equation and conditions (i)–(vii) of Theorem 1 reduce to the corresponding Routh–Hurwitz criterion.

Remark 3 Theorem 1 includes and revises the result of Ponzo [5], and also includes and improves the result of Hu [4] except the restrictions on $a(z,u)$, $b(x,y)$ and $d(x)$, that is, $a(z,u) \leq \alpha + \varepsilon_1$,

$$b(x,y) \leq \mu + \sqrt{\frac{\delta\varepsilon_1}{2\beta}}, \qquad y \int_0^y b_x(x,y) y \, dy \leq -\left(\beta^2\alpha^{-1}\gamma^{-1}\right) y^2$$

and $\gamma - d'(x) \leq \frac{\sqrt{\delta}}{2}$, and the results of Wu and Xiong [8] except the same restrictions on $b(x,y)$.

In the case $p \neq 0$ we have the following result

Theorem 2 *Suppose the following conditions are satisfied:*

(i) *conditions (i)–(vii) of Theorem 1 hold,*

(ii) $|p(t,x,y,z,u)| \leq (A + |y| + |z| + |u|)q(t)$, *where* $q(t)$ *is a non-negative continuous function of* t, *and satisfies*

$$\int_0^t q(s)\,ds \leq B < \infty$$

for all $t \geq 0$, A *and* B *are some positive constants.*

Then for any given finite constants x_0, y_0, z_0 *and* u_0, *there exists a constant* $K = K(x_0, y_0, z_0, u_0)$, *such that any solution* $(x(t), y(t), z(t), u(t))$ *of the system (1.2) determined by*

$$x(0) = x_0, \quad y(0) = y_0, \quad z(0) = z_0, \quad u(0) = u_0$$

satisfies for all $t \geq 0$,

$$|x(t)| \leq K, \quad |y(t)| \leq K, \quad |z(t)| \leq K, \quad |u(t)| \leq K.$$

If p *is a bounded function, then the constant* K *above can be fixed independent of* x_0, y_0, z_0 *and* u_0, *as will be seen from our the following result.*

Theorem 3 *Assume that the conditions (i)–(vii) of Theorem 1 hold, and that* $p(t,x,y,z,u)$ *satisfies*

$$|p(t,x,y,z,u)| \leq A < \infty$$

for all values of t, x, y, z *and* u, *where* A *is a positive constant. Then there exists a constant* K_1 *whose magnitude depends* $\alpha, \beta, \mu, \gamma, \delta, \eta, \varepsilon$ *and* ε_1 *as well as on the functions* a, b, c *and* d *such that every solution* $(x(t), y(t), z(t), u(t))$ *of the system (1.2) ultimately satisfies*

$$|x(t)| \leq K_1, \quad |y(t)| \leq K_1, \quad |z(t)| \leq K_1, \quad |u(t)| \leq K_1.$$

Remark 4 Theorem 2 and Theorem 3 based on the results in ([4], [5], [8]) give additional results to those obtained in ([4], [5], [8]).

The proofs of Theorem 1 and Theorem 2 depend on some certain fundamental properties of a continuously differentiable Lyapunov function $V = V(x,y,z,u)$ defined by:

$$\begin{aligned} V \;=\; & \alpha\gamma \int_0^x d(x)\,dx + \alpha\gamma \int_0^y b(x,y)y\,dy - \left(\frac{\beta\gamma}{2}\right) y^2 + \alpha\beta \int_0^y c(y)\,dy \\ & + \left(\frac{\beta\mu}{2}\right) z^2 + \alpha\beta \int_0^z a(z,0)z\,dz - \left(\frac{\alpha\gamma}{2}\right) z^2 + \left(\frac{\beta}{2}\right) u^2 + \alpha\beta d(x)y \\ & + \beta d(x)z + \beta c(y)z + \alpha\gamma y \int_0^z a(z,0)\,dz + \alpha\gamma yu + \alpha\beta zu. \end{aligned} \tag{2.1}$$

The first property of V is stated in the following.

Lemma 1 *Assume that the conditions of Theorem 1 hold. Then*

(I) $V(x,y,z,u) = 0$ *at* $x^2+y^2+z^2+u^2 = 0.$ (2.2)

(II) $V(x,y,z,w) > 0$ *if* $x^2+y^2+z^2+u^2 > 0;$ (2.3)
$\dot{V}\mid_{(1.2)} \leq 0$ *for all* $t \geq 0.$ (2.4)

(III) *Any of the positive semi-trajectory of the system (1.2) is bounded.*

(IV) *The set* $M = \{(x,y,z,u) : \dot{V} = 0,\ (x,y,z,u) \in R^4\}$, *except* $(x,y,z,u) = 0$, *does not contain the entire positive semi trajectory of the solution of the system (1.2).*

Proof *Part (I):* $V(0,0,0,0) = 0$, since $c(0) = d(0) = 0$. Hence (2.2) is verified.
Rewrite the function $V(x,y,z,u)$ as follows:

$$\begin{aligned} V \;=\; & \left(\frac{\alpha\beta}{2c_1(y)}\right)\left[d(x)+c(y)+\frac{c_1(y)z}{\alpha}\right]^2 \\ & + \left(\frac{\alpha\beta}{2a_1(z,0)}\right)\left[u+za_1(z,0)+\frac{\gamma}{\beta}ya_1(z,0)\right]^2 \\ & + \left(\frac{\beta\mu}{2}\right)z^2 - \left(\frac{\beta c_1(y)}{2\alpha}\right)z^2 - \left(\frac{\alpha\gamma}{2}\right)z^2 \\ & + \alpha\gamma\int_0^y b(x,y)y\,dy - \left(\frac{\beta\gamma}{2}\right)y^2 - \left(\frac{\alpha\gamma^2 a_1(z,0)}{2\beta}\right)y^2 \\ & + \left(\frac{\beta}{2}\right)\left[1-\frac{\alpha}{a_1(z,0)}\right]u^2 + \sum_{i=1}^{3} W_i, \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} W_1 \;&=\; \alpha\gamma\int_0^x d(x)dx - \frac{\alpha\beta d^2(x)}{2c_1(y)}, \\ W_2 \;&=\; \alpha\beta\int_0^y c(y)\,dy - \frac{\alpha\beta c^2(y)}{2c_1(y)}, \\ W_3 \;&=\; \alpha\beta\int_0^z a(z,0)z\,dz - \frac{\alpha\beta a_1(z,0)}{2}z^2. \end{aligned}$$

Part (II): Now we verify (2.3). To do this we have four cases.
(a) Let $y \neq 0$, $z \neq 0$. From (iv) of Theorem 1 it follows that

$$W_1 \geq \alpha\gamma\int_0^x d(x)\,dx - \frac{\alpha d^2(x)}{2} \geq \alpha\int_0^x d(x)[\gamma - d'(x)]\,dx \geq 0.$$

Now note that

$$yc(y) \equiv \int_0^y c(y)\,dy + \int_0^y c'(y)y\,dy.$$

Therefore,

$$W_2 = \alpha\beta \int_0^y c(y)\,dy - \frac{\alpha\beta c(y)}{2} = \frac{\alpha\beta}{2}\int_0^y [c_1(y) - c'(y)]y\,dy \geq -\left(\frac{\alpha\beta\eta}{4}\right) y^2$$

by (vi). From the identity

$$\int_0^z za(z,0)\,dz \equiv z\int_0^z a(z,0)\,dz - \int_0^z za_1(z,0)\,dz$$

we find

$$\begin{aligned} W_3 \;&=\; \alpha\beta\int_0^z a(z,0)z\,dz - \frac{\alpha\beta}{2}z\int_0^z a(z,0)\,dz \\ &= \frac{\alpha\beta}{2}\int_0^z [a(z,0) - a_1(z,0)]z\,dz \geq -\left(\frac{\alpha\beta\varepsilon}{4}\right) z^2 \end{aligned}$$

by (vi) of Theorem 1. On gathering the estimates for W_1, W_2 and W_3 into (2.5), we have that

$$\begin{aligned} V \;\geq\; & \alpha\int_0^x d(x)[\gamma - d'(x)]\,dx + \left(\frac{\alpha\beta}{2a_1(z,0)}\right)\left[u + za_1(z,0) + \frac{\gamma}{\beta}ya_1(z,0)\right]^2 \\ & + \left(\frac{\alpha\beta}{2c_1(y)}\right)\left[d(x) + c(y) + \frac{c_1(y)z}{\alpha}\right]^2 + \left(\frac{\beta\mu}{2}\right) z^2 \\ & - \left(\frac{1}{2\alpha}\right)\left[\beta c_1(y) + \alpha^2\gamma + \frac{\alpha^2\beta\varepsilon}{2}\right] z^2 + \alpha\gamma\int_0^y b(x,y)y\,dy \\ & - \left(\frac{\beta\gamma}{2}\right) y^2 - \left(\frac{\gamma}{2\beta}\right)\left[\alpha\gamma a_1(z,0) + \frac{\alpha\beta^2\eta}{2\gamma}\right] y^2 \\ & + \left(\frac{\beta}{2}\right)\left[1 - \frac{\alpha}{a_1(z,0)}\right] u^2. \end{aligned} \tag{2.6}$$

Now consider the terms

$$W_4 = \left(\frac{\beta\mu}{2}\right) z^2 - \left(\frac{1}{2\alpha}\right)\left[\beta c_1(y) + \alpha^2\gamma + \frac{\alpha^2\beta\varepsilon}{2}\right] z^2$$

and

$$W_5 = \alpha\gamma\int_0^y b(x,y)y\,dy - \left(\frac{\beta\gamma}{2}\right) y^2 - \left(\frac{\gamma}{2\beta}\right)\left[\alpha\gamma a_1(z,0) + \frac{\alpha\beta^2\eta}{2\gamma}\right] y^2$$

which are contained in (2.6).

By using the assumptions (i), (v), (vi) of Theorem 1 and the mean value theorem (for derivative), we find

$$\begin{aligned} W_4 \;&=\; \left(\frac{1}{2\alpha}\right)\left[\alpha\beta\mu - \beta c'(\theta_1 y) - \alpha^2\gamma - \frac{\alpha^2\beta\varepsilon}{2}\right] z^2 \\ &\geq \left(\frac{1}{2\alpha}\right)\left[\alpha\beta\mu - \beta c'(\theta_1 y) - \alpha\gamma a(z,u) - \frac{\alpha^2\beta\varepsilon}{2}\right] z^2 \\ &\geq \left(\frac{1}{2\alpha}\right)\left[\delta - \frac{\alpha^2\beta\varepsilon}{2}\right] z^2 > 0, \end{aligned}$$

where $0 \le \theta_1 \le 1$. Similarly, from (iii), (v), (vi) of Theorem 1 and the mean value theorem (for integral), we obtain

$$\begin{aligned} W_5 &\ge \left(\frac{\gamma}{2\beta}\right)\left[\alpha\beta\mu - \beta^2 - \alpha\gamma a_1(z,0) - \frac{\alpha\beta^2\eta}{2\gamma}\right] y^2 \\ &= \left(\frac{\gamma}{2\beta}\right)\left[\alpha\beta\mu - \beta^2 - \alpha\gamma a(\theta_2 z,0) - \frac{\alpha\beta^2\eta}{2\gamma}\right] y^2 \\ &\ge \left(\frac{\gamma}{2\beta}\right)\left[\delta - \frac{\alpha\beta^2\eta}{2\gamma}\right] y^2 > 0, \end{aligned}$$

where $0 \le \theta_2 \le 1$. On substituting the estimate for W_4 and W_5 into (2.6) we have

$$\begin{aligned} V \ge{}& \alpha\int_0^x d(x)[\gamma - d'(x)]\,dx + \left(\frac{\alpha\beta}{2a_1(z,0)}\right)\left[u + za_1(z,0) + \frac{\gamma}{\beta} y a_1(z,0)\right]^2 \\ &+ \left(\frac{\alpha\beta}{2c_1(y)}\right)\left[d(x) + c(y) + \frac{c_1(y)z}{\alpha}\right]^2 + \left(\frac{1}{2\alpha}\right)\left[\delta - \frac{\alpha^2\beta\varepsilon}{2}\right] z^2 \\ &+ \left(\frac{\gamma}{2\beta}\right)\left[\delta - \frac{\alpha\beta^2\eta}{2\gamma}\right] y^2 + \left(\frac{\beta}{2}\right)\left[1 - \frac{\alpha}{a_1(z,0)}\right] u^2 > 0. \end{aligned}$$

(b) Let $y^2 + z^2 = 0$. Then it follows from (2.5) that

$$V \ge \alpha\gamma\int_0^x d(x)\,dx + \left(\frac{\beta}{2}\right) u^2 > 0 \quad \text{if } x^2 + u^2 > 0.$$

(c) Let $y \ne 0$, $z = 0$. Similarly, it is easy to see that

$$\begin{aligned} V \ge{}& \alpha\int_0^x d(x)[\gamma - d'(x)]\,dx \\ &+ \left(\frac{\alpha\beta}{2a_1(0,0)}\right)\left[u + \frac{\gamma}{\beta} y a_1(0,0)\right]^2 + \left(\frac{\alpha\beta}{2c_1(y)}\right)[d(x) + c(y)]^2 \\ &+ \left(\frac{\gamma}{2\beta}\right)\left[\delta - \frac{\alpha\beta^2\eta}{2\gamma}\right] y^2 + \left(\frac{\beta}{2}\right)\left[1 - \frac{\alpha}{a_1(0,0)}\right] u^2 > 0. \end{aligned}$$

(d) Let $y = 0$, $z \ne 0$. It is clear from (a) that

$$\begin{aligned} V \ge{}& \alpha\int_0^x d(x)[\gamma - d'(x)]\,dx \\ &+ \left(\frac{\alpha\beta}{2a_1(z,0)}\right)[u + za_1(z,0)]^2 + \left(\frac{\alpha\beta}{2c_1(0)}\right)\left[d(x) + \frac{c_1(0)z}{\alpha}\right]^2 \\ &+ \left(\frac{1}{2\alpha}\right)\left[\delta - \frac{\alpha^2\beta\varepsilon}{2}\right] z^2 + \left(\frac{\beta}{2}\right)\left[1 - \frac{\alpha}{a_1(z,0)}\right] u^2 > 0 \end{aligned}$$

by (2.5). Because of the estimates given by (a)–(d) we get the desired result (2.3).

From (2.1) and (1.2) it is trivial that the time derivative of V as follows:

$$\begin{aligned}\dot{V} = & -\alpha\beta\left[\frac{c(y)}{y}\frac{\gamma}{\beta} - d'(x)\right]y^2 \\ & - \left[\alpha\beta b(x,y) - \beta c'(y) - \alpha\gamma\left(\frac{1}{z}\right)\int_0^z a(z,0)\,dz\right]z^2 \\ & - \beta\left[a(z,u) - \alpha\right]u^2 - \beta\left[b(x,y) - \mu\right]zu - \beta\left[\gamma - d'(x)\right]yz \\ & + \alpha\gamma y\int_0^y b_x(x,y)y\,dy \\ & - \alpha\gamma\left[a(z,u) - a(z,0)\right]yu - \alpha\beta\left[a(z,u) - a(z,0)\right]zu.\end{aligned}$$

Hence the assumptions (i)–(v) of Theorem 1 and the mean value theorem (for the integral) show that

$$\begin{aligned}\dot{V} \le & -\left[\alpha\beta\mu - \beta c'(y) - \alpha\gamma a(\theta_3 z,0)\right]z^2 \\ & - (\beta\varepsilon_1)u^2 - \beta\left[b(x,y) - \mu\right]zu - \beta\left[\gamma - d'(x)\right]yz \\ & + \alpha\gamma y\int_0^y b_x(x,y)y\,dy \\ & - \alpha\gamma\left[a(z,u) - a(z,0)\right]yu - \alpha\beta\left[a(z,u) - a(z,0)\right]zu, \quad (0 \le \theta_3 \le 1), \\ \le & -\left(\frac{3\beta\varepsilon_1}{4}\right)u^2 - \left(\frac{\delta}{2}\right)z^2 - \left(\frac{3\beta^2}{4}\right)y^2 - W_6 - W_7 - W_8, \qquad (2.7)\end{aligned}$$

where

$$\begin{aligned}W_6 &= \left(\frac{\delta}{4}\right)z^2 + \beta\left[b(x,y) - \mu\right]zu + \left(\frac{\beta\varepsilon_1}{4}\right)u^2, \\ W_7 &= \left(\frac{\beta^2}{4}\right)y^2 + \beta\left[\gamma - d'(x)\right]yz + \left(\frac{\delta}{4}\right)z^2, \\ W_8 &= \alpha\gamma\left[a(z,u) - a(z,0)\right]yu + \alpha\beta\left[a(z,u) - a(z,0)\right]zu.\end{aligned}$$

From (iii) of Theorem 1

$$W_6 \ge \left(\frac{\delta}{4}\right)z^2 - \beta\left[b(x,y) - \mu\right]|zu| + \left(\frac{\beta\varepsilon_1}{4}\right)u^2 = \left[\frac{\sqrt{\delta}}{2}z \pm \frac{\sqrt{\beta\varepsilon_1}}{2}u\right]^2 \ge 0.$$

Similarly, by (iv) of Theorem 1, we find

$$W_7 \ge \left(\frac{\beta^2}{4}\right)y^2 - \beta\left[\gamma - d'(x)\right]|yz| + \left(\frac{\delta}{4}\right)z^2 = \left[\frac{\beta}{2}y \pm \frac{\sqrt{\delta}}{2}z\right]^2 \ge 0.$$

The assumption (vii) of Theorem 1 (for $u \ne 0$) also shows that

$$W_8 = \alpha\left[\gamma y a_u(z,\theta_4 u) + \beta z a_u(z,\theta_4 u)\right]u^2 \ge 0, 0 \le \theta_4 \le 1,$$

but $W_8 = 0$, when $u = 0$. Hence $W_8 \ge 0$ for all y, z and u.

On combining the estimates for W_6, W_7 and W_8 into (2.7) we find

$$\dot{V} \leq -\left(\frac{3\beta\varepsilon_1}{4}\right)u^2 - \left(\frac{\delta}{2}\right)z^2 - \left(\frac{3\beta^2}{4}\right)y^2.$$

This completes the proof of Part (II).

The proofs of Part (III) and Part (IV) follow the lines indicated in [4], except some minor modification. And hence the proof is omitted.

This completes the proof of the lemma. □

The proof of Theorem 1 From Lemma 1, we see that the function $V(x, y, z, u)$ is a Lyapunov function for the system (1.2). Hence, the zero solution of the system (1.2) is asymptotically stable (see [8]).

This completes the proof. □

The proof of Theorem 2 The proof of this theorem is similar to that of Theorem 2 of Tunc [7] and hence is omitted.

Finally, the actual proof of Theorem 3 will rest mainly on the existence of a piecewise continuously differentiable function $V_1 = V_1(x, y, z, u)$ satisfying

$$V_1(x, y, z, u) \geq -D \quad \text{for all } (x, y, z, u), \tag{2.8}$$

$$V_1(x, y, z, u) \to \infty \quad \text{as } x^2 + y^2 + z^2 + u^2 \to \infty; \tag{2.9}$$

and also such that the limit

$$\dot{V}_1^+(t) = \limsup_{h\to 0+}\left[\frac{V_1(x(t+h), y(t+h), z(t+h), u(t+h)) - V_1(x(t), y(t), z(t), u(t))}{h}\right] \tag{2.10}$$

exists corresponding any solution $(x(t), y(t), z(t), u(t))$ of the system (1.2), and satisfies

$$\dot{V}_1^+(t) \leq -1 \quad \text{if } x^2(t) + y^2(t) + z^2(t) + u^2(t) \geq D_1,$$

where D and D_1 are certain positive constants to be determined in the proof.

Once the existence of such a V_1 is established an appeal to Yoshizawa's argument (see [2]) concludes the proof of Theorem 3.

We define the required V_1 as follows:

$$V_1 = V_0 + V, \tag{2.11}$$

where

$$V_0(x, u) := \begin{cases} x \operatorname{sgn} u, & \text{if } |u| \geq |x| \\ u \operatorname{sgn} x, & \text{if } |u| \leq |x| \end{cases} \tag{2.12}$$

and V is defined by (2.1).

The property of $\dot{V}_1^+$ is required and is stated in Lemma 2.

Lemma 2 *Subject to the conditions of Theorem 3, the function V_1 defined in (2.11) satisfies the properties in (2.8), (2.9) and (2.10).*

Proof Let (x, y, z, u) be any solution of the system (1.2). From (2.12) we obtain $|V_0(x,u)| \leq |u|$ for all x and u. It follows that $|V_0(x,u)| \geq -|u|$ for all x and u. Now, V here is the same as the function V defined by (2.1). Since V is positive definite, then it has infinite inferior limit and infinitesimal upper limit, that is, there exists a positive constant τ such that

$$V(x, y, z, u) > \tau(x^2 + y^2 + z^2 + u^2).$$

From these estimates for V_0 and V we get the estimate for V_1 as

$$V_1 > \tau(x^2 + y^2 + z^2 + u^2) - 2\,|u| = \tau(x^2 + y^2 + z^2) + \tau\left(|u| - \frac{1}{\tau}\right)^2 - \frac{1}{\tau}.$$

So it is evident that (2.8) and (2.9) are verified, where $D = \frac{1}{\tau}$.

Next, in accordance with the representation $V_1 = V + V_0$ we have a representation $v_1 = v + v_0$. Hence, the function $v_1 = v_1(t)$ can be defined by $v_1(t) = V_1(x(t), y(t), z(t), u(t))$. Then, the existence of $\dot{v}_1^+$, that is,

$$\dot{v}_1^+(t) = \limsup_{h \to 0+}\left[\frac{v_1(t+h) - v_1(t)}{h}\right]$$

is quite immediate, since v has continuous first partial derivatives and v_0 is easily shown to be locally Lipschitizian in x and u so that the composite function $v_1 = v + v_0$ is at the least locally Lipschitizian in x, y, z and u. Subject to the assumptions of the theorem an easy calculation from (2.11) and (1.2) shows that

$$\dot{v}_1^+ = \dot{v} + \dot{v}_0^+ \leq -\left(\frac{3\beta\varepsilon_1}{4}\right)u^2 - \left(\frac{\delta}{2}\right)z^2 - \left(\frac{3\beta^2}{4}\right)y^2 + D_2(|y| + |z| + |u|), \text{ if } |u| \geq |x|$$

or

$$\dot{v}_1^+ = \dot{v} + \dot{v}_0^+ \leq -\left(\frac{3\beta\varepsilon_1}{4}\right)u^2 - \left(\frac{\delta}{2}\right)z^2 - \left(\frac{3\beta^2}{4}\right)y^2 - d(x)sgnx + |c(y)| + D_3(1 + |y| + |z| + |u|), \quad \text{if } |u| \leq |x|.$$

The following arguments are similar to those in [3] and hence we omit the details of the proof. The proof of this lemma is now complete. □

The proof of Theorem 3 By considering the results obtained in Lemma 2, the usual Yoshizawa-type argument (see the result established in [2]) applied to (2.8), (2.9) and (2.10) would then show that, for any solution (x, y, z, u) of the system (1.2), we have

$$|x(t)| \leq K_1, \quad |y(t)| \leq K_1, \quad |z(t)| \leq K_1, \quad |u(t)| \leq K_1,$$

for all sufficiently large t, which proves the theorem.

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References

[1] Barbashin, E. A.: Liapunov's Function. *Science*, 1970.

[2] Chukwu, E. N.: *On the boundedness of a certain fourth-order differential equation.* J. London Math. Soc. (2) **11**, 3 (1975), 313–324.

[3] Ezeilo, J. O. C. , Tejumola, H. O.: *On the boundedness and the stability properties of solutions of certain fourth order differential equation.* Ann. Mat. Pure. Apl. (IV) **95** (1973), 131–145.

[4] Hu, C. Y.: *The stability in the large for certain fourth order differential equations.* Ann. Differential Equations **8**, 4 (1992), 422–428.

[5] Ponzo, P. J.: *On the stability of certain nonlinear differential equations.* IEEE Trans. Automatic Control **10** (1965), 470–472.

[6] Reissig, R., Sansone, G., Conti, R.: Nonlinear Differential Equations of Higher Order. *Noordhoff, Groningen*, 1974.

[7] Tunç, C.: *A note on the stability and boundedness results of solutions of certain fourth order differential equations.* Appl. Math. Comp. **155**, 3 (2004), 837–843.

[8] Wu, X., Xiong, K.: *Remarks on stability results for the solutions of certain fourth-order autonomous differential equations.* Internat. J. Control. **69**, 2 (1998), 353–360.

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